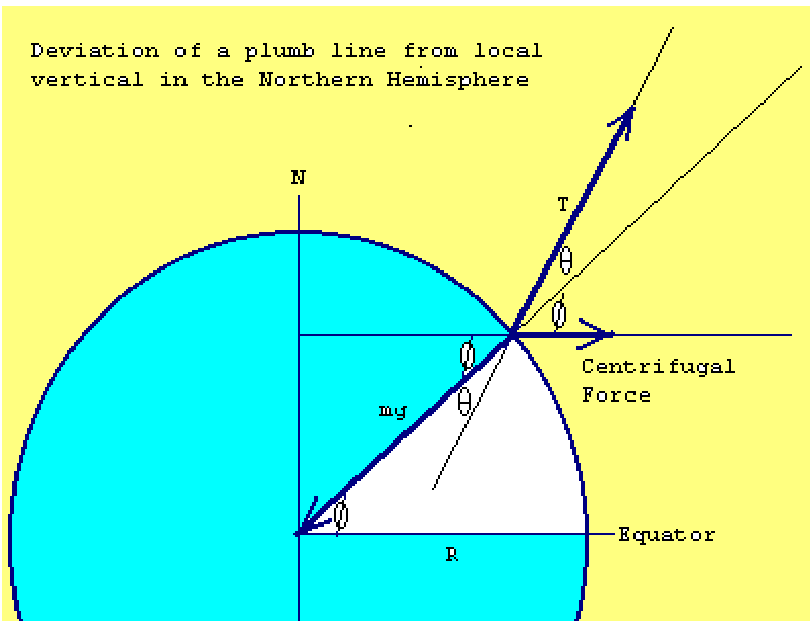
Working in the Rotating Reference Frame of the Earth

Deviation of a Plumb Line from Vertical as a Function of Latitude

[From *Physics by Example*, problem #34]

Find the analytical expression for the deflection (*deviation in angle*) of a stationary plumb line from the true local vertical at latitude ϕ. Calculate the maximum value of this deflection from vertical and the latitude where this maximum value occurs. Assume the Earth is a uniform sphere of radius 6,400 km and the acceleration due to gravity is constant and equal to 9.81 m/s^2.



In the rotating (*accelerating*) reference frame affixed to the Earth we observe the centrifugal force, which is equal and opposite to the centripetal force observed when we view the Earth from an inertial reference frame outside the Earth frame. We know we are accelerating because our velocity vector constantly changes direction (*in any inertial reference frame*) as the Earth rotates on its axis, even though the speed (*the magnitude of the velocity vector*) does not change (*in either reference frame*).

Since the plumb line is at rest in the reference frame of the Earth the Coriolis force is zero. The Coriolis force is another of those mystery "forces" that only shows up because we are in a non-inertial reference frame. It is the "force" that causes airplanes to drift eastward as they fly northward in the northern hemisphere.

The vector sum of the three forces, mg, tension and centrifugal force, must add up to zero since the bob on the plumb line is at equilibrium in the reference frame of the Earth. To begin the analysis, we first resolve the three vectors into their vertical and horizontal components.

$$T_{vertical} + F_{c,\ vertical} = F_{Gravity}$$

$$T \cos\theta + m\omega^2 R \cos^2\phi = mg \qquad (\textit{sum of vertical components = 0})$$

$$T_{horizontal} = F_{c,horizontal} + \text{(gravity has no horizontal component)}$$

$$T \sin\theta = m\omega^2 R \cos\phi \sin\phi \qquad (\textit{sum of horizontal components = 0})$$

This gives us two equations in two unknowns, θ and T. We don't need T so our first step is to eliminate it from the equations so we can solve for θ. We begin by eliminating the tension. There is one simple and obvious way to eliminate T.

Divide the first equation by the second. This eliminates both T and mass from the resulting equation.

$$\frac{T \cos\theta}{T \sin\theta} = \frac{m\,(g - \omega^2 R \cos^2\phi)}{m\,(\omega^2 R \cos\phi \sin\phi)} = \cot\theta$$

After canceling both T and m we get the intermediate result for cot θ, as follows:

$$\cot\theta = \frac{(g - \omega^2 R \cos^2\phi)}{(\omega^2 R \cos\phi \sin\phi)}$$

Remember that we are looking for θ not cot θ, so our work is not yet complete. The criterion for finding an analytical expression is not "Can I plug it into my calculator?" The criterion is "Have I found an analytical expression for the required variable?" Better yet is to also have the independent variable appearing only once in the expression so that it could, if necessary, be solved graphically.

We can simplify our work on this expression in two ways:

I - Note the following numerical values:

$\omega = 2\pi/24$ hrs $= 2\pi/86400$ sec $= 1/13{,}750$

$\omega^2 = (1/13{,}750)^2 = 5.289\text{e-}9\ s^{-2}$

R = 6.4e3 km = 6.4e6 m

$\omega^2 R = 0.03385\ m/s^2$ (*Of course, the* $\cos^2\phi$ *makes it even smaller.*)

This shows that g (=9.81) is almost 300 times larger than $\omega^2 R$. We would simply say that g is much, much greater than $\omega^2 R$ (i.e., $g \gg \omega^2 R$). When adding or subtracting these two numbers we can ignore the smaller term in the sum. (*This would not be true when multiplying or dividing, however.*) This approximation introduces an error of about one-third of one percent into the cotangent calculation. We expect that it will make a correspondingly small difference in the final θ.

II - There is a famous approximation we can also use here. It says that when working with small angles (*in radians now, not in degrees, where small means less than about 5 to 10 degrees; i.e. less than about 0.08727 to 0.174533 radians*), we can substitute for sin x and tan x, as follows; (*You should try these for yourself with some small angles. Your calculator must be in radians mode for this to work.*)

$$\sin x \approx x$$

$$\tan x \approx x$$

Our angle θ in this case is going to turn out to be very small, so this approximation works fine. Substituting both approximations into the cotangent equation above yields:

$$\cot\theta = \frac{1}{\tan\theta} \approx \frac{1}{\theta} \approx \frac{g}{\omega^2 R \cos\phi \sin\phi} \text{, and therefore}$$

$$\theta \approx \frac{\omega^2 R}{g} \cos\phi \sin\phi \approx \frac{\omega^2 R}{2g} \sin(2\phi)$$

{*Where we've also used the trigonometric identity: 2 sin x cos x = sin (2x)*}

This is the expression we've been seeking. Now we're ready for the numerical results. First, we can find the latitude where the maximum value of θ occurs by simply looking at the equation and making some observations.

Clearly, the maximum value of θ occurs when sin (2ϕ) reaches its maximum value. That occurs when sin (2ϕ) equals one (*because all other values in this expression are constants*). The sine function equals one when the angle, 2ϕ in this case, equals π/2 radians or 90 degrees. Therefore, θ reaches its maximum value when ϕ = 45 degrees (*that means 45 degrees north latitude or 45 degrees south latitude*).

The maximum value of the deviation itself can now be calculated from the expression for θ derived above using the values for ω, R, and g already discussed, and the value of ϕ just found, as follows:

$$\theta = 1.7\text{e-}3 \text{ radians} = 0.099 \text{ degrees} \approx 0.1 \text{ degree}$$

This treatment ignores the non-spherical shape of the planet. That also has an effect; about one-half the size of this one though not always in the same direction.