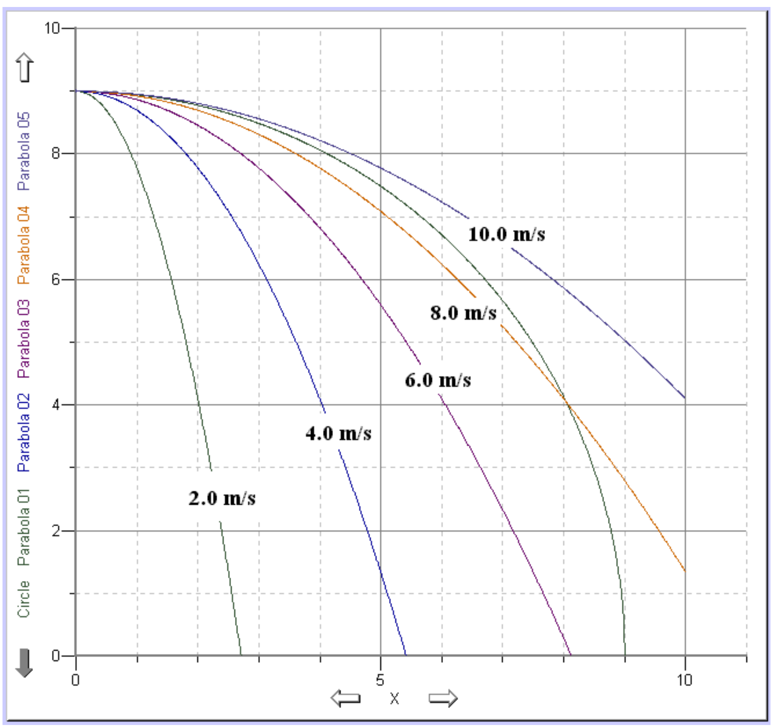
Projectile Motion Equations

Ball Kicked Horizontally Off a Spherical Rock

I. A ball is sitting on top of a spherical rock of radius 9.00 m. A soccer player kicks the ball along a trajectory that is initially horizontal. Find the minimum initial velocity required for the ball to reach the level ground beyond the edge of the spherical rock.

Obs. #1. The ball falls along a parabolic trajectory. The tangent at the apex of the parabola is horizontal. In spite of that the ball does not get above the rock unless it has a certain minimum speed. This happens because over the initial part of its range the parabola represents a steeper curve that the circle.

Here are some representative parabolas for our case where the radius of the rock is 9 m.



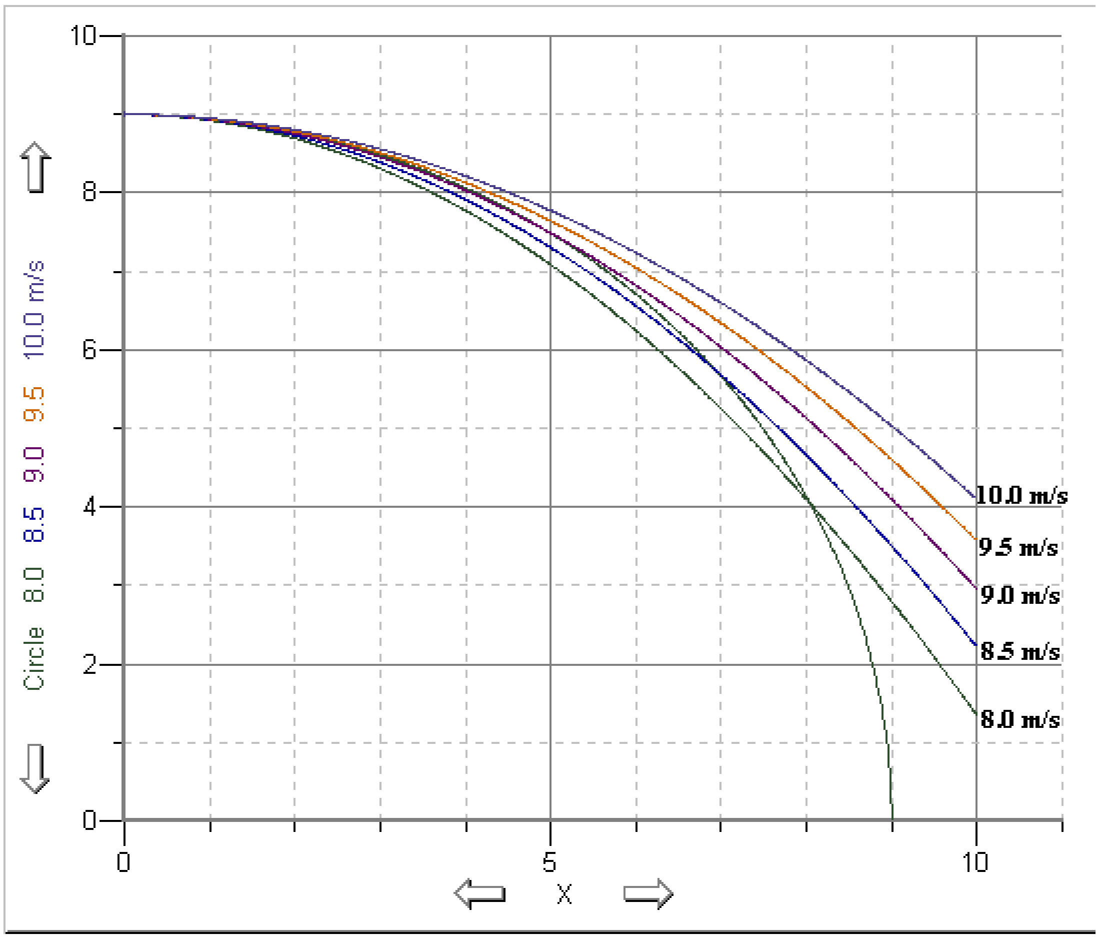
The trajectory curves are based on the equation we developed in another handout for y as a function of x.

It looks like the minimum initial velocity is going to be between 8 and 10 meters per second for a rock with a 9 m radius; and probably closer to 10 m/s than to 8 m/s.

Its going to be difficult to tell what the minimum is without looking a bit more closely at the range between 8 and 10 m/s.

Just for the fun of it, let’s check that region out a little more carefully.

Here is a graph showing trajectories between 8 m/s and 10 m/s. The trajectories actually travel through the rock and emerge at different distances from the starting position. Note that as the speed increases the ball emerges closer to the origin. We have to find the speed where it emerges at the origin.



The slowest plotted trajectory that never enters the rock might be 9.5 m/s. The answer has to be close to 9.5 m/s. Provisionally, we can take 9.5 m/s as our answer.

Still, knowing the answer doesn't really help us solve the general problem, but it will allow us to check any answer we get to see if it is reasonable. As part of our investigation for this question, let's look at relationships among y, x and r a little more closely.

Obs. #2. To get some additional feel for the form of the solution, we'll investigate the form of the equation that includes the launch angle of 0° and the initial position, $y_0 = r$. Thus,

$$y - y_0 = y - r = -\tfrac{1}{2}g\, x^2 / v_0^2$$

Furthermore, we know that y = 0 when the ball reaches the ground at x_{MIN}. Therefore,

$$r = \tfrac{1}{2}g\, x_{MIN}^2 / v_0^2$$

$$2r/g = x_{MIN}^2 / v_0^2$$

The value of x_{MIN} when y = 0 must be greater than r. Let $x_{MIN} = nr$, where n > 1. Then

$$2r/g = n^2r^2 / v_0^2$$

$$2/g = n^2r / v_0^2$$

$$v_0^2 = (n^2r\, g) / 2 = (n^2/2)\, r\, g$$

The result would be particularly simple <u>IF</u>, for example, the value of n happened to be $2^{1/2}$.

In that event, $v_0 = (r\, g)^{1/2}$

(That last guess is motivated by the observation that for our problem $(r\, g)^{1/2} = 9.396$ m/s. *As that is between 8 and 10 m/s, and close to 9.5 m/s, it has to be considered a contender. In fact, given that the range of the answer has to be between* 8 *and* 10 m/s, *we can conclude that* n *must lie in the range between 1.2040 and 1.5051. Given that the square root of 2 also lies in this range, it seems like an obvious candidate for* n. *)*

IF $n = 2^{1/2}$, then the value of $x_{MIN} = 2^{1/2}\, r$. The distance from the edge of the sphere to the landing spot is then

$$x_{MIN} - r = 2^{1/2}\, r - r = (2^{1/2} - 1)\, r.$$

This kind of playing with the problem is highly recommended. It provides unexpected insights, at times, and can often be very enlightening about the best route to the solution of the problem. Also, it is just plain fun sometimes to toy around with the problem trying different approaches. Sometimes it requires this type of circuitous route to see your way through a problem. Don't be afraid to play a little.

In this problem, for example, playing around tells you that the ½ is going to disappear, somehow. The x is going to disappear, too. The solution needs a lot of symmetry to get rid of all these pieces. We are going to need something special with this one. We know it starts out messy. How can we clean it up?

Ans. Start with the obvious relationship between the y-values

The initial height of the ball is $y_0 = r$

The height of the semicircle at any point is $y_C = (r^2 - x^2)^{½}$

The drop distance of the ball is $y_B = ½g\, x^2 / v_0^2$

The fundamental inequality defining this problem is

$$y_0 - y_C - y_B = r - (r^2 - x^2)^{½} - ½g\, x^2 / v_0^2 > 0$$

We are going to use the Binomial Expansion to simplify the square-root term in this expression.

$$(r^2 - x^2)^{½} = r\,[1 - (x^2/ r^2)]^{½} = r\,[1 - ½\,(x^2/ r^2)]$$

To justify using the Binomial Expansion we need to justify the assumption that

$$(x^2/r^2) << 1.$$

The insight into this route to the solution comes from the graphs generated under Obs. #1. If you look at the second graph, you will see that the point where the ball escapes from the sphere moves toward smaller x-values as the velocity increases. That means we need to concentrate on fitting the initial velocity during the earliest part of the flight of the ball, ie at the smallest x-values. When the criterion is satisfied for the smallest values of x, the ball will miss the rock. Therefore

$$y_0 - y_C - y_B = r - r\,[(1 - ½\,(x^2/ r^2)] - ½g\, x^2 / v_0^2 > 0$$

$$r - r + ½\, r\,(x^2/ r^2) > ½g\, x^2 / v_0^2$$

$$½\, r\,(x^2/ r^2) > ½g\, x^2 / v_0^2$$

$$r\,(1/ r^2) > g / v_0^2$$

$$v_0^2\, r\,(1/ r^2) > g$$

$$v_0^2(1/ r) > g$$

$$v_0^2 > rg$$

$$v_0 > (r\, g)^{½}$$

This is the solution guessed in Obs. #2. A sphere 9 m in radius requires an initial velocity of 9.396 m/s. The distance the ball lands beyond the edge of the sphere is then

$$x_{MIN} - r = 2^{½}\, r - r = (2^{½} - 1)\, r = 0.414r = 3.73 \text{ m}$$