Projectile Motion Equations - Y as a function of x, θ, g;

the Range equation; the Maximum Height equation; and the ratio of Maximum Height to Range.

I. A projectile subject only to the force of gravity is launched at an angle θ with respect to the horizontal plane. The initial velocity is v_0. Start with the equations of motion for the x and y components of the motion and derive one equation for y(x, θ, g).

Ans. Start with these two equations of motion

$$x - x_0 = v_{x0}\,t = (v_0 \cos\theta)\,t$$

and

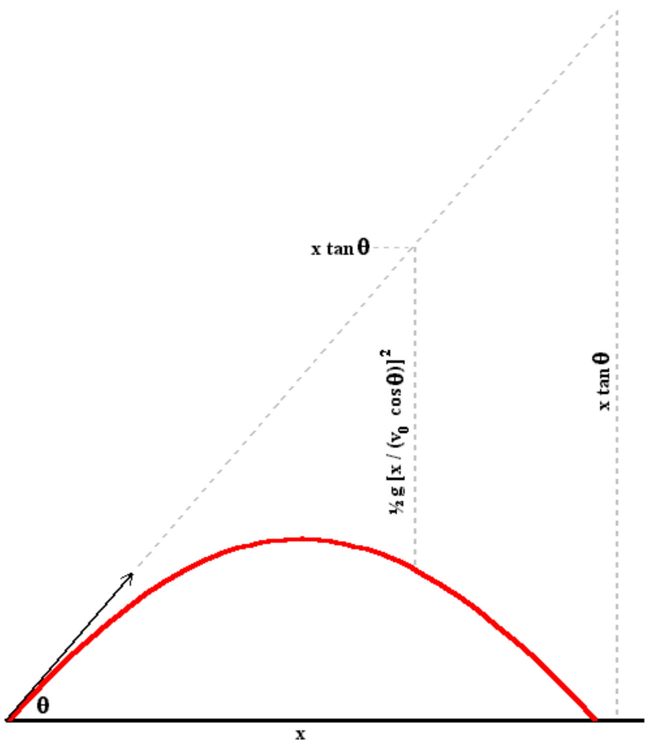
$$y - y_0 = v_{yo}\,t + \tfrac{1}{2} a t^2 = (v_0 \sin\theta)\,t - \tfrac{1}{2} g t^2$$

Assume that the initial position is at the origin of our reference frame. The equations thus become

$$x = (v_0 \cos\theta)\,t$$

and

$$y = v_0 (\sin\theta)\,t - \tfrac{1}{2} g t^2$$



To eliminate the t, solve the first equation for t and substitute into the second.

$$t = x / (v_0 \cos\theta)$$

Thus,

$$y = v_0 (\sin\theta)\,x / (v_0 \cos\theta) - \tfrac{1}{2} g\,[x / (v_0 \cos\theta)]^2$$

$$y = x\,[(\sin\theta) / (\cos\theta)] - \tfrac{1}{2} g\,[x / (v_0 \cos\theta)]^2$$

$$y = x \tan\theta - \tfrac{1}{2} g x^2 / (v_0 \cos\theta)^2$$

Here is the spreadsheet formula that reproduces this equation in Column C in Excel.

=A7*TAN(F2*PI()/180) – (0.5*C4*A7^2)/(C2*COS(F2*PI()/180))^2

*x * tan(θ) – ½ * g * x^2 / (v_0 * cos(θ))^2*

Where v_0 is at C2; θ is at F2; g is at C4; x is in column A and Δx is at F4.

II. Assume that a projectile is launched over a flat horizontal plane. The projectile will land at the same elevation from which it was launched. This landing distance is called the range of the projectile.

Find an equation for the range of this projectile in terms of v_0 and θ.

Ans. When the projectile reaches its range, its elevation will again be zero. We can use the previous result to find the value of x_{RANGE} by setting the y value on the left side of the equation to zero. Thus,

$$0 = x_{RANGE} \tan\theta - ½\, g\, x_{RANGE}^2 / (v_0 \cos\theta)^2$$

$$\cos^2\theta \cdot x_{RANGE} \tan\theta = \cos^2\theta \cdot ½\, g\, x_{RANGE}^2 / (v_0 \cos\theta)^2$$

$$x_{RANGE} \sin\theta \cos\theta = ½\, g\, x_{RANGE}^2 / (v_0)^2$$

$$2 \sin\theta \cos\theta = g\, x_{RANGE} / (v_0)^2$$

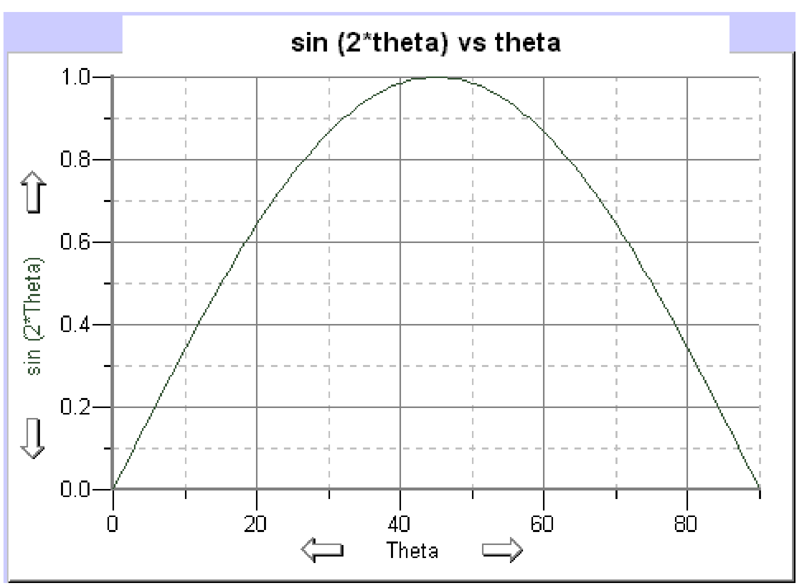
$$\sin 2\theta = g\, x_{RANGE} / (v_0)^2$$

$$x_{RANGE} = [\sin 2\theta]\, (v_0)^2 / g$$

For every angle less than 45º there is an angle greater than 45º that has the same value of sin 2θ. The normal sine function goes only from 0 up to 1 when the angle changes between 0º and 90º. The sin 2θ function goes from 0 up to 1 and back down to 0 again as the angle changes between 0º and 90º. Here is the graph of sin 2θ vs θ.

There are two values of θ that define each possible range. The exception is $\theta = 45$º, which is the unique angle that produces the maximum range.

Notice that the function is symmetric about the 45º line.



III. Find the equations that give

a) the x-position where the projectile reaches this maximum height

b) the maximum height reached by a projectile and

c) the ratio of the maximum height to the range of the projectile

all as function of v_0 and θ.

Ans. The first is the easiest. The maximum altitude is reached when the projectile is halfway to its range. Therefore, the x-position of maximum altitude, x_{MAX} is given by

a) $x_{MAX} = ½\, x_{RANGE} = ½\, [\sin 2\theta]\, (v_0)^2/g = [\sin\theta \cos\theta]\, (v_0)^2/g$

From this result we can easily calculate the altitude, y_{MAX}, at x_{MAX}.

b) $y_{MAX} = x_{MAX} \tan\theta - ½\, g\, x_{MAX}{}^2 / (v_0 \cos\theta)^2$

$y_{MAX} = [\sin\theta / \cos\theta]\, [\sin\theta \cos\theta]\, (v_0)^2/g - ½\, g\, x_{MAX}{}^2 / (v_0 \cos\theta)^2$

$y_{MAX} = [\sin\theta]^2\, (v_0)^2/g - ½\, g\, (x_{MAX})^2 / (v_0 \cos\theta)^2$

$y_{MAX} = [\sin\theta]^2\, (v_0)^2/g - ½\, g\, ([\sin\theta \cos\theta]\, (v_0)^2/g)^2 / (v_0 \cos\theta)^2$

$y_{MAX} = [\sin\theta]^2\, (v_0)^2/g - ½\, g\, ([\sin^2\theta \cos^2\theta]\, (v_0)^4) / (g^2\, v_0{}^2 \cos^2\theta)$

$y_{MAX} = ½\, [\sin^2\theta]\, (v_0)^2/g$

$y_{MAX} = ½\, [v_0 \sin\theta]^2/g = v_{y0}{}^2 / (2g)$

c) The ratio of y_{MAX} / x_{RANGE} is given by

$y_{MAX} / x_{RANGE} = [½\, [v_0 \sin\theta]^2/g] / [[2 \sin\theta \cos\theta]\, (v_0)^2/g]$

$y_{MAX} / x_{RANGE} = [½\, [v_0{}^2 \sin^2\theta]/g] / [v_0{}^2\, [2 \sin\theta \cos\theta]/g]$

$y_{MAX} / x_{RANGE} = [½\, [\sin^2\theta]] / [[2 \sin\theta \cos\theta]]$

$y_{MAX} / x_{RANGE} = [½\, [\sin\theta]] / [[2 \cos\theta]]$

$y_{MAX} / x_{RANGE} = [½ \sin\theta] / [2 \cos\theta]$

$y_{MAX} / x_{RANGE} = ¼ \tan\theta$

(Bonus Result: $y_{MAX} / (½\, x_{RANGE}) = y_{MAX} / x_{MAX} = ¼ \tan\theta / ½ = ½ \tan\theta$)