Vector Multiplication - Cross Product or Vector Product

The vector product of two vectors is a vector quantity whose direction is perpendicular to both vectors (i.e. perpendicular to the plane defined by the two vectors when they are placed tail to tail). It is defined as follows:

$$\mathbf{a} \times \mathbf{b} = |a||b| \sin \theta$$

where θ is the angle between the vectors
when they are placed tail-to-tail

or as

$$\mathbf{a} \times \mathbf{b} = [x_a \, \boldsymbol{i} + y_a \boldsymbol{j} + z_a \, \boldsymbol{k}] \times [x_b \, \boldsymbol{i} + y_b \boldsymbol{j} + z_b \, \boldsymbol{k}] =$$

$$+ x_a x_b \, \boldsymbol{i}\times\boldsymbol{i} + x_a y_b \, \boldsymbol{i}\times\boldsymbol{j} + x_a z_b \, \boldsymbol{i}\times\boldsymbol{k}$$
$$+ y_a x_b \, \boldsymbol{j}\times\boldsymbol{i} + y_a y_b \, \boldsymbol{j}\times\boldsymbol{j} + y_a z_b \, \boldsymbol{j}\times\boldsymbol{k}$$
$$+ z_a x_b \, \boldsymbol{k}\times\boldsymbol{i} + z_a y_b \, \boldsymbol{k}\times\boldsymbol{j} + z_a z_b \, \boldsymbol{k}\times\boldsymbol{k} =$$

$$+ 0 + x_a y_b \, \boldsymbol{k} + x_a z_b \, (-\boldsymbol{j})$$
$$+ y_a x_b \, (-\boldsymbol{k}) + 0 + y_a z_b \, \boldsymbol{i}$$
$$+ z_a x_b \, \boldsymbol{j} + z_a y_b \, (-\boldsymbol{i}) + 0 =$$

$$(y_a z_b - z_a y_b) \, \boldsymbol{i} + (z_a x_b - x_a z_b) \, \boldsymbol{j} + (x_a y_b - y_a x_b) \boldsymbol{k}$$

The cross product is order dependent. In combinations involving both the dot product and the cross product, the cross products must be performed first. One way to remember the multiplication table for the cross products of the unit vectors is

$$\boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} = \boldsymbol{k}\times\boldsymbol{k} = 0$$

$$\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k} \qquad \boldsymbol{j}\times\boldsymbol{i} = -\,\boldsymbol{k}$$
$$\boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i} \qquad \boldsymbol{k}\times\boldsymbol{j} = -\,\boldsymbol{i}$$
$$\boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j} \qquad \boldsymbol{i}\times\boldsymbol{k} = -\,\boldsymbol{j}$$

The other way to evaluate the cross product is to remember this alternate definition in terms of a determinate. Evaluate the determinate and you will have the vector that results from the cross product. The result is the same either way.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{vmatrix} = (y_a z_b - z_a y_b) \, \boldsymbol{i} - (x_a z_b - z_a x_b) \, \boldsymbol{j} + (x_a y_b - y_a x_b) \boldsymbol{k}$$

The definition in terms of magnitudes and the angle between the vectors works best when working with 2-dimensional vectors. In three dimensions it is often very difficult to find the angle between the two vectors. In three dimensions, or even in two dimensions, it is usually best to work with the component forms of the vectors and use the component definition of the cross product.

Relationships that obey the cross product produce the maximum interaction when the two vectors are perpendicular and no interaction when the vectors are parallel to each other. We'll use this form of vector multiplication when we examine the interactions between moving charged particles and magnetic fields.