



**1. continued.** Use the shadow length to determine the angle the sun made with the vertical at the moment of your measurement. The length of the shadow in millimeters divided by the length of the meter stick in millimeters (*1000 mm*) equals the tangent of the shadow angle. Calculate the inverse tangent to find the shadow angle in degrees and radians to 6 digits (*4 SD plus two extra digits to minimize the opportunities for round-off errors when using the trigonometric functions*).

Shadow Angle =  $c_{\text{Measured}} = \tan^{-1}(\tan c) = \tan^{-1}(L(\text{mm}) / 1000 \text{ mm})$ , therefore

$c_{\text{Measured}} = \underline{\hspace{2cm}}^{\circ} = \underline{\hspace{2cm}}$  radians

(Use the arctangent)      (radians = degrees •  $\pi / 180^{\circ}$ )

**2.** The next thing you need to know is the school's geographical location. You can obtain this from either an atlas or your instructor.

Latitude<sub>school</sub> =  $\underline{\hspace{1cm}}^{\circ} \underline{\hspace{1cm}}' \underline{\hspace{1cm}}''$ , Longitude<sub>school</sub> =  $\underline{\hspace{1cm}}^{\circ} \underline{\hspace{1cm}}' \underline{\hspace{1cm}}''$   

Degrees    Min    Sec    N/S
Degrees    Min    Sec    E/W

Convert the latitude and longitude of the school from degree-minute-second notation to decimal degrees. (*North latitude is positive. West longitude is positive. Keep 6 digits in the decimal forms.*)

Latitude<sub>school</sub> =  $\underline{\hspace{2cm}}$ °, Longitude<sub>school</sub> =  $\underline{\hspace{2cm}}$ °  

(From instructor)
(From instructor)

**3.** Request that the instructor provide you with the distance from the zenith point of the Sun to the school and the geographical coordinates of the Sun's zenith point at the exact moment of your measurement. (*Recall that Eratosthenes needed both the shadow angle and distance to the zenith point of the Sun. Fortunately for him, his zenith point was well known so He could measure the distance at his leisure. We are not as lucky.*)

Record the information here. Using the coordinates of the school, the Sun, and a little spherical trigonometry you will verify this distance to the zenith point for yourself in section 6.

$D_{ZS}$  = (distance from the zenith point of the Sun to the school) =  $\underline{\hspace{2cm}}$  km =  $\underline{\hspace{2cm}}$  m  
(From the computer simulation)

Enter the coordinates of the zenith point of the Sun at the moment of your measurement.

Latitude<sub>Sun</sub> =  $\underline{\hspace{1cm}}^{\circ} \underline{\hspace{1cm}}' \underline{\hspace{1cm}}''$ , Longitude<sub>Sun</sub> =  $\underline{\hspace{1cm}}^{\circ} \underline{\hspace{1cm}}' \underline{\hspace{1cm}}''$   

(From the computer simulation)
(From the computer simulation)

Convert the latitude and longitude of the zenith point from degree-minute-second notation to decimal degrees. (*For latitude, north is + and south is -. For longitude west is + and east is -.*)

Latitude<sub>Sun</sub> =  $\underline{\hspace{2cm}}$ °, Longitude<sub>Sun</sub> =  $\underline{\hspace{2cm}}$ °

4. Begin by recalling Eratosthenes’ method. Visualize the school and the zenith points on a globe. By definition, the sun casts no shadow at the zenith point. You measured the shadow angle at the school. We’ve used a modern method to calculate the distance to the zenith point (*Because it would be impossible to pace off the distance yourselves*), but you can use a very old fashioned one to calculate the circumference of the Earth.

Eratosthenes’ own method relies on the evident fact that the following ratios are equal. Solving this equation, as Eratosthenes did, allows you to determine the circumference of the Earth. Make sure all distances are in the same units. The meter is the preferred unit for distance measures, such as circumference and zenith distance.

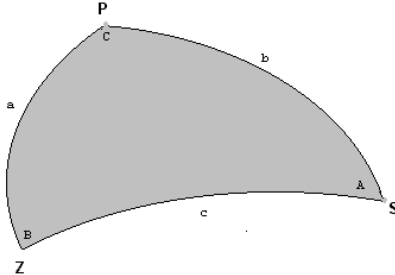
$$\frac{360^\circ}{\text{shadow angle}} = \frac{\text{circumference of the Earth}}{\text{zenith distance}}$$

5. Solve the equation used by Eratosthenes to calculate your estimate for circumference of the Earth,  $C_{\text{Earth}}$ . Then use  $C_{\text{Earth}}$  to estimate the radius of the Earth,  $R_{\text{Earth}}$ . Use this estimate of the radius in section 6.

$C_{\text{Earth}} = D_{\text{ZS}} \cdot 360^\circ / c_{\text{Measured}}^\circ = \underline{\hspace{2cm}} \text{ m} = \underline{\hspace{2cm}} \text{ km}$   
 $R_{\text{Earth}} = C_{\text{Earth}} / (2\pi) = \underline{\hspace{2cm}} \text{ m} = \underline{\hspace{2cm}} \text{ km}$

It is often claimed that Eratosthenes measured the size of the Earth to within 1% of the true value. There is a discussion of this claim at the end of the lab. In the Analysis section you will compare your own values of the circumference and radius of the Earth to the true values and calculate the percent error in your measurement of the size of the Earth.

**6. Spherical Trigonometry** - If you assume that the Earth is a perfect sphere (*which it is not, but it's pretty close, so we will make the assumption anyway*), then it is relatively easy to find the distance between two points on its surface from their geographic coordinates and the radius. The location of the zenith point of the Sun is south of schools in the United States, but this discussion applies to any school or location north of the Sun's current zenith point.



The school is at **S** and the Sun's zenith point is at **Z**. Construct a spherical triangle by going due north from the zenith point to the North Pole, at point **P**. This line is automatically a great circle since it follows a longitude line. Then proceed along another longitude line from point **P** to the school, at **S**. What you want to find is the great-circle distance on the third side of the triangle from point **Z** to point **S**,  $D_{ZS}$ , the distance from the zenith point of the Sun to the school. To do that we first need to find the shadow angle **c**.

We will use the Spherical Law of Cosines recast into a form where it is written in terms of the latitude (Lat) and longitude (Lon) of the points **S** (*School*) and **Z** (*Zenith Point*). In the following formulae all the distances are angular distances. Angles **A**, **B** and **C** (*see the figure*) are measured on the surface of the Earth using the equivalent of a protractor. The angles **a**, **b** and **c** are measured as though you are standing at the center of the Earth and sighting along radii of the sphere to the end points of each arc.

In fact, we've already measured angle **c**. That is the angle we called the shadow angle and designated  $c_{\text{Measured}}$  in section 1 on page 2. In this section we will estimate the value of angle **c** from our knowledge of the latitude and longitude of the school and the zenith point of the Sun. This new estimate of the angle **c** will be designate as  $c_{\text{estimated}}$ .

It is easiest to make this calculation in stages, as follows: (*Double and triple-check these calculations!*)

$$a = 90^\circ - \text{Latitude}_{\text{Sun}} \text{ (Latitude is positive if North, negative if South)} = \underline{\hspace{2cm}}^\circ$$

$$b = 90^\circ - \text{Latitude}_{\text{School}} \text{ (Latitude is positive if North, negative if South)} = \underline{\hspace{2cm}}^\circ$$

$$C = | \text{Longitude}_{\text{Sun}} - \text{Longitude}_{\text{School}} | \text{ (Longitude + if West, and - if East)} = \underline{\hspace{2cm}}^\circ$$

From the Spherical Law of Cosines we can calculate the cosine of  $c_{\text{estimated}}$ ,

$$\cos c_{\text{estimated}} = [\cos a \times \cos b] + [\sin a \times \sin b \times \cos C] = \underline{\hspace{2cm}}$$

Therefore, the angle we will call  $c_{\text{estimated}}$  is given by;  $c_{\text{estimated}} = \cos^{-1}(\cos c_{\text{estimated}})$

$$c_{\text{estimated}} = \underline{\hspace{2cm}}^\circ$$

$$c_{\text{estimated}} = \underline{\hspace{2cm}} \text{ radians}$$

(radians = degrees  $\times \pi / 180$ )

The computer gave us the zenith distance,  $D_{ZS}$ . Here we will estimate  $D_{ZS}$  from all our measurements. The estimated zenith distance, designated  $D_{ZSE}$ , is given by  $D_{ZSE} = \text{radius of Earth} \times c_{\text{estimated}}$  (in radians)

$$D_{ZSE} = R_{\text{Earth}} \times c_{\text{estimated}} = \underline{\hspace{2cm}} \text{ m} \times \underline{\hspace{2cm}} \text{ radians} = \underline{\hspace{2cm}} \text{ m}$$

Distance (Z to S) = (Radius of the Earth-Section 5)  $\times$  (Shadow Angle,  $c_{\text{estimated}}$ ) = (Distance from Z to S)

$$D_{ZSE} = R_{\text{Earth}} \times c_{\text{estimated}} = \underline{\hspace{2cm}} \text{ km} \times \underline{\hspace{2cm}} \text{ radians} = \underline{\hspace{2cm}} \text{ km}$$

Distance (Z to S) = (Radius of the Earth-Section 5)  $\times$  (Shadow Angle,  $c_{\text{estimated}}$ ) = (Distance from Z to S)

**Analysis:** In the Analysis Section we must accomplish the following goals:

- Compare your calculated circumference to the standard value. The accuracy of your answer ultimately depends on the accuracy of your shadow length measurement.
- Compare the distances between the zenith point and the school calculated using the instructor's computer program vs using spherical trigonometry and your estimated radius.

We will frequently use %Error or %Uncertainty, and %Difference to quantify such comparisons. In the case of the circumference, where the true value is known, you will quantify the difference as a %Error.

$$\begin{aligned}\%Error &= 100\% \cdot |(measured\ value) - (true\ value)| / (true\ value) \\ &= 100\% \cdot |uncertainty\ in\ measured\ value| / (true\ value)\end{aligned}$$

In the case of the zenith distance, where a very good but less certain reference value is available, we will quantify the difference as a %Uncertainty or %UNC, for short.

$$\begin{aligned}\%UNC &= 100\% \cdot |(measured\ value) - (reference\ value)| / (reference\ value) \\ &= 100\% \cdot |uncertainty\ in\ measured\ value| / (reference\ value)\end{aligned}$$

*(To find a worthwhile reference value look for a table in your physics textbook. Failing that, Google the web looking for the circumference of the Earth, or you may try some other written reference source. You may need to calculate the circumference from a radius. The same %Error applies to your estimated radius since it derives from your estimated circumference.) List your **source** for the true value of the circumference or radius, the **value**, and **units**. (Try to find an original source in m or km.)*

Source: \_\_\_\_\_ page: \_\_\_\_\_

$C_{Earth}$  = true circumference = \_\_\_\_\_ m = \_\_\_\_\_ km

$R_{Earth}$  = true radius = \_\_\_\_\_ m = \_\_\_\_\_ km

First, you need to report on the **accuracy** of your calculated circumference. The true value of the circumference is known. **Accuracy is the measure of how close your answer is to the true value.** Accuracy can be reported as %Error.

%Error in the measured circumference = \_\_\_\_\_ % (SD)

Next, you need to report the %Uncertainty in your calculated Zenith Distance. Use the instructor's value of the zenith distance,  $D_{ZS}$ , as the reference value. Calculate the %Uncertainty in your estimate of the zenith distance,  $D_{ZSE}$ , calculated on page 4.

%Uncertainty in your estimated Zenith Distance ( $D_{ZSE}$ ) = \_\_\_\_\_ % (SD)

Analysis: Continued

A meter stick allows you, in principle, to make measurements to the nearest millimeter and estimate to the nearest tenth of a millimeter. What %Uncertainty in the shadow length would you expect to find if the uncertainty in your measured shadow length was exactly  $\pm 1/10^{\text{th}}$  of a millimeter? Or exactly  $\pm 1$  mm? Or exactly  $\pm 5$  mm? Or exactly  $\pm 10$  mm? Or exactly  $\pm 20$  mm? Or exactly  $\pm 30$  mm?

*(Use the measured shadow length, L, is the reference length.)*

*(The equation, given the uncertainty is: **%Uncertainty = 100% × uncertainty/reference**)*

%Uncertainty ( $\pm 0.0001$  m) = \_\_\_\_\_ % (SD)

%Uncertainty ( $\pm 0.001$  m) = \_\_\_\_\_ % (SD)

%Uncertainty ( $\pm 0.005$  m) = \_\_\_\_\_ % (SD)

%Uncertainty ( $\pm 0.01$  m) = \_\_\_\_\_ % (SD)

%Uncertainty ( $\pm 0.02$  m) = \_\_\_\_\_ % (SD)

%Uncertainty ( $\pm 0.03$  m) = \_\_\_\_\_ % (SD)

The %Uncertainty in your Zenith Distance is due almost entirely to the uncertainty in your measured shadow length.

Interpolate from the list above using your %Uncertainty in the circumference of the Earth, and make an educated guess of the %Uncertainty in your measured shadow length.

%Uncertainty (in the shadow length) = \_\_\_\_\_ %

From this you can determine the uncertainty in the measurement of the shadow length.

Uncertainty (shadow length) = [%Uncertainty/100%] × shadow length (mm) = \_\_\_\_\_ mm

List here the experimental factors, in addition to the meter stick's scale, that contributed to the experimental uncertainty in your measured shadow length. Suggest ways of modifying the procedure or decreasing the difficulty to decrease the uncertainty in future measurements of the shadow length?

**Sample Calculations:** Show one complete calculation of each type. Examples include: one conversion of an angle in hours, minutes and seconds to decimal degrees; one calculation of %Error; one conversion of angle in degrees to radians; etc. Do your best to identify one example if every type of calculation. Label each sample calculation clearly.

*(Attach additional sheet of paper, as necessary, to complete this index of calculations. The instructor's list contains 12 different types of calculations performed in completing this lab. You need to find at least 10.)*

**Note:** Your calculated %Error and %Uncertainty must have no more significant digits than the number of significant digits in your measurement. The measurement of the shadow length has 4 SD according to the special rule on page 1. Therefore, your answers for these values must have 4 SD, also. To avoid round-off errors in these calculations, you should keep track of one or two additional digits in the intermediate values.

## The “REAL” Story Behind Eratosthenes

There seems to be some confusion about Eratosthenes’ result. While I am not prepared to say with certainty what he found, I think we can safely predict what he would have been able to find for the circumference of the Earth.

Eratosthenes got an answer for the circumference somewhere near the true value of 40,075.16 km. That’s all we know. The question before us is, how good a result could he have gotten? There were some problems he could not have known about or controlled.

First, Syene (*modern Aswan*) isn’t exactly on the Tropic of Cancer. Second, it is not directly south of Alexandria, as the simplest interpretation requires.

Let’s assume he measured the correct value for the shadow angle on June 21. At Alexandria this would be, approximately, the Latitude of Alexandria minus the Latitude of the Tropic of Cancer ( $31^{\circ} 11' 24'' - 23^{\circ} 27' 18''$ ),  $\Delta L = 7^{\circ} 44' 06'' (= 7.735^{\circ})$ . After a small adjustment for atmospheric refraction, we can predict he should have measured an angle of  $7.603338^{\circ}$ . (*Of course, he did not work in degrees at all; the 360° circle having, so far, utterly failed to get itself invented. But you get the idea. He would have used the fraction of a full circle.*)

Then there is the distance problem. Aswan is  $3^{\circ}$  of longitude East of Alexandria. Since the surveyors walked from Alexandria to Aswan, they walked too far. The true distance as the crow flies from Aswan to Alexandria is 884.947 km. Along the Nile River it is even farther as the river does not meander a straight path between the two cities. We’ll assume the surveyors took all that into account and got the equivalent of 884.947 km.

On the other hand, the shortest distance from Alexandria to the Tropic of Cancer, based on a due south course, is only

$$7.603338^{\circ} \cdot 60 \cdot 1.15 \cdot 5280 \cdot 12 \cdot 2.54 \div 100 \div 1000 = 844.310654 \text{ km}$$

$$(^{\circ}) \cdot (\text{arc-min}/^{\circ}) \cdot (\text{statute mi}/\text{arc-min}) \cdot (\text{ft}/\text{mi}) \cdot (\text{in}/\text{ft}) \cdot (\text{cm}/\text{in}) \div (\text{cm}/\text{m}) \div (\text{m}/\text{km}) = \text{km}$$

The best result that Eratosthenes could have found, therefore, seems to be

$$884.947 \cdot 360^{\circ} / 7.603338^{\circ} = 41900.14 \text{ km}$$

That is a 4.55% Error compared to 40,075.16 km. To the extent that the surveyors got the distance to Syene wrong, his answer could be off in either direction. That seems quite good, but there is more to this problem.

It should be worth comparing this to the best answer we can get using his method. Using the uncorrected shadow angle of  $7.603338^{\circ}$  and a distance to the Tropic of Cancer based on modern maps, we can calculate that his reference value would be,

$$844.310654 \cdot 360 / 7.603338^{\circ} = 39976 \text{ km}$$

Compared to this reference value his best result is off by 4.8 %. Again depending on the surveyor’s methodology, his error could have been higher or lower.

Additional error could result from measuring the angle on the wrong day, which would always decrease his estimate of the size of the Earth. We will assume he made no measurement or mathematical errors.

Another problem we have in trying to understand, today, what Eratosthenes did, then, is that we don't know the exact length of the stade, also known as the stadium; his unit of length.

The stadium is 125 roman-paces. The roman-pace is 5 roman-feet. The problem is that the length of the roman-foot is uncertain. Best guesses are that it might be 296 mm or maybe 295.7 mm. It could also be something else entirely.

Eratosthenes 5000 stadia, usually listed as his distance between Alexandria and Syene, turns out to be 924.0625 km. If that reflects anything like his true result, then his error is even larger. He would have gotten, at best, a circumference of

$$924.0625 \cdot 360^\circ / 7.603338^\circ = 43752.17 \text{ km}$$

Compared to the 39976 km listed above, the error in his result is 9.45%. Compared with the reference value of 40,075.16 km listed earlier, his error is 9.2%.

Only some combination of errors or misunderstanding could have given Eratosthenes the 1% error sometimes claimed for his result. Some reports claim that he rounded the measured distance up to 5000 stadia. If that is true, then his result might have been even closer to the correct modern value, if he hadn't rounded the distance. If the number of stadia had been about 10% smaller or if the stade used in the measurements was 10% shorter than assumed here, he would have been right on the target.

Eratosthenes wasn't the only Greek who tried to measure the Earth. About a century later, Posidonius recreated this feat, using the star Canopus as his light source and the cities of Rhodes and Alexandria as his baseline. Although his technique was sound, he had the wrong value for the distance between Rhodes and Alexandria. Measuring distances over water was beyond the technology of his time.

His circumference came out far too small. However, Ptolemy recorded this smaller figure in his geography treatise. Renaissance explorers looking for a quicker way to the Indies seized upon the smaller value reported by Ptolemy. Had Ptolemy chosen Eratosthenes' larger figure instead Posidonius' smaller value, Columbus might never have received the funding to sail west.

**These problems are now on WebAssign – Study them.****Exercises:** Show your work. (*Express answers to the correct number of significant digits.*)

1. What is the volume in cubic inches of a box 25.5-cm by 32 cm by 89.2 cm?
2. What is the volume in cubic centimeters of a ball with a radius of 12 inches? Look up the formula in the Appendix.
3. What is the surface area in square feet of one side of a CD 12.3 cm in diameter, assuming the center hole is 1.00 cm in diameter?
4. What is the volume in liters of a metal drum 46 cm in diameter and 83.2 cm tall?
5. What is the area of an ellipse 45.88-cm tall by 12.113-cm wide?
6. What is the perimeter in meters of a quadrilateral with sides of 3.4 cm, 124 mm, 1.33 m, & 82 cm?
7. What is the great arc surface distance from Pretoria, South Africa to Washington, D.C. (*Find an atlas and determine the latitude and longitude of these two cities yourself. Then use the law of cosines for spherical triangles to calculate the distance in meters between them. Don't forget that south latitudes are negative and east longitudes are negative.*)

**Interesting facts:**

- 1) The navigator's problem is a different from the one you performed in this lab. The navigator's job is to find the ship's geographic coordinates from the known coordinates of the zenith point and the measured shadow angle. Approached from that perspective the best that the navigator can accomplish is to locate the ship somewhere on a large circle centered on the zenith point.

To find the ship's position, the navigator must create two such circles. If you lay out two circles on the surface of a sphere you will find that they intersect at two places. If the navigator has any idea where the ship is located, then it should be easy to decide at which of two points the ship is located.

There are several ways to get the two circles. The navigator can use the Sun at two different times. At night two or more stars can be used by taking nearly simultaneous sights. Even the planets can be used and sometimes they can even be used during daylight. The Moon can also be used during the day and at night. If the navigator has a means of locating the zenith point of any celestial object and can see that object, then it can be used to locate the ship.

- 2) The zenith point of the Sun moves  $360^\circ$  in 24 hrs on average. Thus it moves  $15^\circ$ , or one time zone, each hour;  $\frac{1}{4}^\circ$  or 15' (arc-minutes) each minute;  $\frac{1}{4}'$  or 15" (arc-seconds) each second.
- 3) Most sailors know the saying: "a minute's a mile." That is, one arc-minute of latitude (= 1') translates into 1 nautical mile distance. A nautical mile = 1.15 statute miles = 1.8507 km.
- 4) It takes the Sun's zenith point only 4 sec to move one nautical mile (that is  $\sim 1035$  mph or  $\sim 900$  knots).
- 5) Say "arc-minute" and "arc-second" to distinguish the angles from times in "minutes" and "seconds".