

## INTRODUCTION

Before we can understand the positions and motions of celestial bodies, it is necessary to have an understanding of *spherical geometry*. Spherical geometry is the study of geometry on the surface of a sphere and is the spherical analogue of the planar geometry you will have studied at school. In this series of lectures we will explore some of the basic concepts of spherical geometry. We will then show how these concepts can be applied by solving some everyday problems to do with the measurement of position and distance on the surface of the Earth.

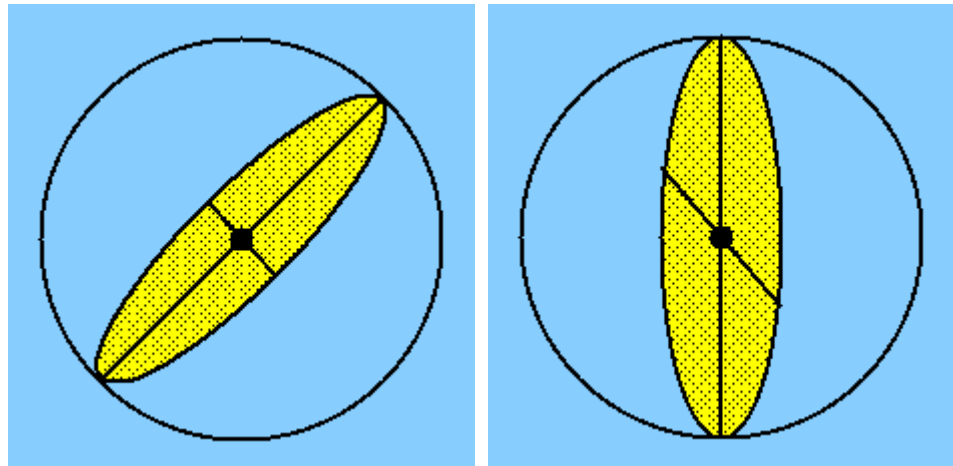
## THE GEOMETRY OF THE SPHERE

We all know that the shortest path between two points on a plane is a straight line. On the surface of a sphere, however, there are no straight lines. The shortest path between two points on the surface of a sphere is the arc of a *great circle* passing through the two points. For a pair of points on a sphere, only one great circle passes through both points. A great circle is defined to be the intersection with a sphere with a plane containing the center of the sphere. If the plane does not contain the center of the sphere, its intersection with the sphere is known as a *small circle*. In more everyday language, if we take an apple, assumed to be a sphere, and cut it in half, we slice along a great circle. If we make a mistake, miss the center and hence cut the apple into two unequal parts, we will have sliced through a small circle. Figure 1 illustrates the concepts of great and small circles.

**Figure 1:**  
The Black Dot  
Represents the  
Center of the Earth

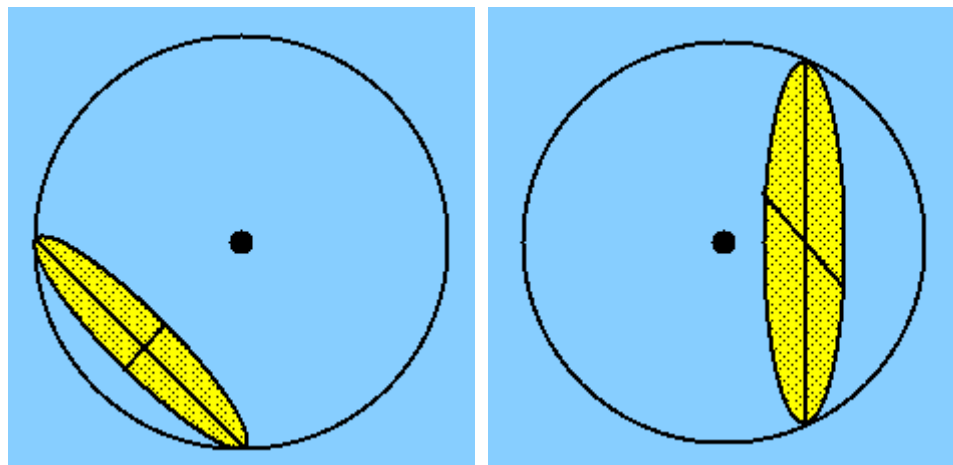
*Upper Panels*

Great Circles



*Lower Panels*

Small Circles



## SPHERICAL TRIANGLES

Another concept we should all be familiar with from planar geometry is the triangle. If we wish to connect three points on a plane using the shortest possible route, we would draw straight lines and hence create a triangle. By analogy, if we wish to connect three points on the surface of a sphere using the shortest possible route, we would draw arcs of great circles and hence create a *spherical triangle*. To avoid ambiguities, a triangle drawn on the surface of a sphere is only a spherical triangle if it has *all* of the following properties:

- The three sides are all arcs of great circles.
- Any two sides are together longer than the third side.
- The sum of the three angles is greater than  $180^\circ$ .
- Each individual spherical angle is less than  $180^\circ$ .

Hence, in Figure 2, triangle  $PABD$  is *not* a spherical triangle (as the side  $AB$  is an arc of a small circle), but triangle  $PCDP$  is a spherical triangle (as the side  $CD$  is an arc of a great circle). You can see that the above definition of a spherical triangle also rules out the "triangle"  $PCEDP$  as a spherical triangle, as the vertex angle  $P$  is greater than  $180^\circ$  and the sum of the sides  $PC$  and  $PD$  is less than side  $CED$ .

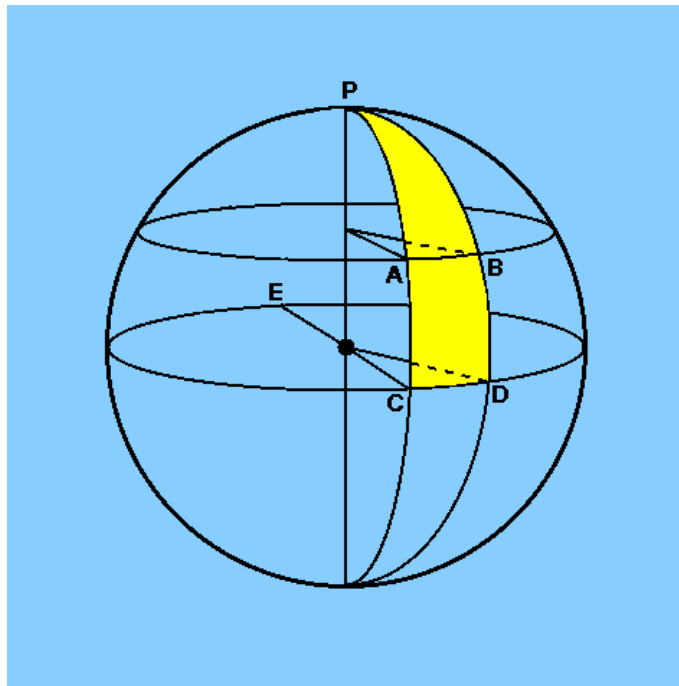
**Figure 2:**

Triangle  $PCDP$  is a spherical Triangle

Triangle  $PABP$  is not a Spherical Triangle because side  $AB$  is not an arc that is part of a Great Circle

Triangle  $PCEDP$  is not a Spherical Triangle because the angle at  $P$  is greater than  $180^\circ$  **and** because the length of side  $CED$  is greater than the sum of the other two sides,  $PC$  and  $PD$ .

It is important to realize that the lengths of the sides of a spherical triangle are expressed in angular measure. This follows from the fact that the length of an arc  $s$  which



subtends an angle  $\theta$  at the center of a circle of radius  $r$  is given by

$$s = (\theta / 360) \cdot 2\pi r$$

where  $\theta$  is measured in degrees and  $s$  is given in the same length units used for  $r$ .

Recalling that  $2\pi$  radians =  $360^\circ$ , allows us to rewrite that expression as

$$s = \theta r$$

where  $\theta$  is now measured in radians and  $s$  is still given in whatever length units are used for  $r$ . Given that spherical triangles are all made up of great circle arcs, and all great circle arcs on a sphere are of the same radius, it is convenient to take the radius of the sphere as unity and then write

$$s = \theta$$

which says that that we can express the length of a side of a spherical triangle as the angle (in radians) it subtends at the center of the unit sphere. If  $s$  is one-quarter of the circumference of a great circle, then

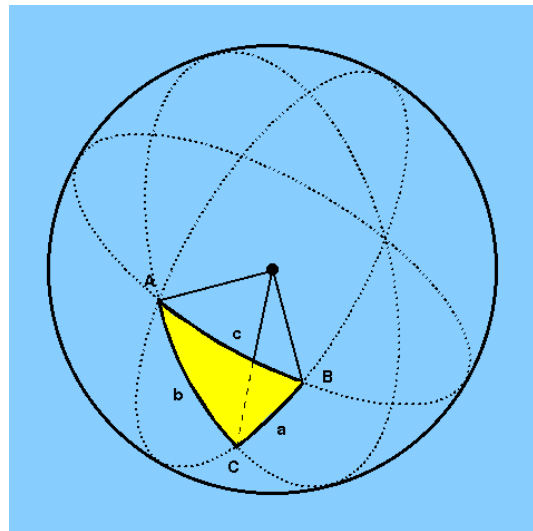
$$s = (2\pi / 4) \text{ radians} = (\pi / 2) \text{ radians (there is no ambiguity if it is then expressed, for convenience, as } 90^\circ)$$

## SPHERICAL TRIGONOMETRY

To perform calculations with planar triangles it is necessary to use the formulae of planar trigonometry, which you will have met before. Similarly, to perform calculations with spherical triangles it is necessary to use the formulae of *spherical trigonometry*, which are given below. Figure 3 shows a spherical triangle, formed by three intersecting great circles, with arcs of length  $(a, b, c)$  and vertex angles of  $(A, B, C)$ .

**Figure 3:**

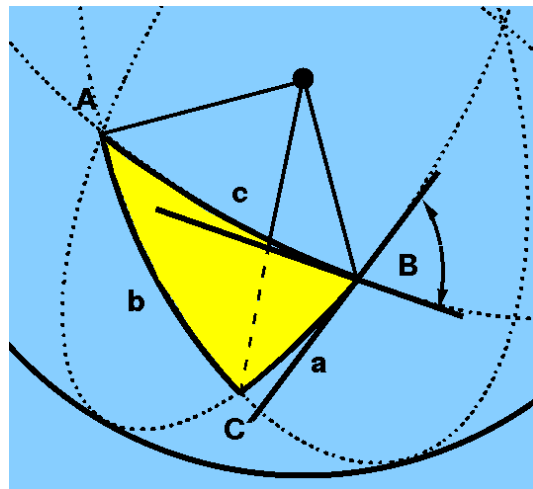
A spherical triangle with arcs of length  $(a, b, c)$  and vertex angles of  $(A, B, C)$ .



Note that the angle between two sides of a spherical triangle is defined as the angle between the *tangents* to the two great circle arcs, as shown in Figure 4 for vertex angle  $B$ .

**Figure 4:**

The vertex angle  $B$  is defined as the angle between the tangents to the two great circle arcs,  $a$  and  $c$ .



The arc lengths ( $a, b, c$ ) and vertex angles ( $A, B, C$ ) of the spherical triangle in Figure 4 are related by the following formulae:

**The Spherical Law of Sines formula:**  
 $(\sin a / \sin A) = (\sin b / \sin B) = (\sin c / \sin C)$

**The Spherical Law of Cosines formula:**  
 $\cos a = \cos b \cos c + \sin b \sin c \cos A$

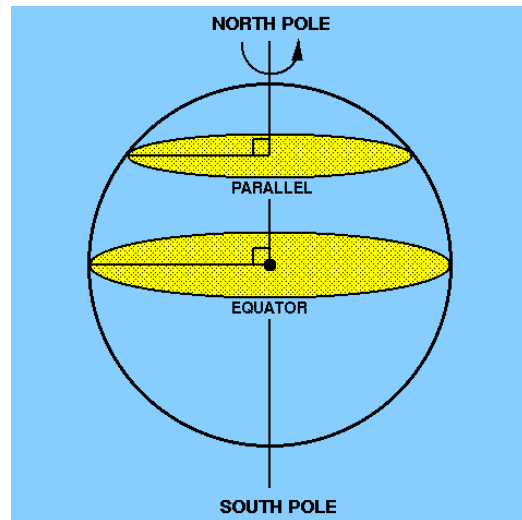
These are the spherical analogues of the Law of Sines and the Law of Cosines of planar trigonometry. Their proofs are given in many textbooks and reference books, along with a host of other formulae which you will not need to know - almost all of the simple problems you will encounter in spherical geometry can be solved using just the sine and/or cosine formulae given above. The trick is to know which triangle to draw - once you have done this, the solution is easy, as shown in the example problems following the next section.

## POSITION ON THE EARTH'S SURFACE

To see how some of the spherical geometry described above can be put to good use, let's look at an everyday example - the measurement of position and distance on the surface of the Earth. The rotation of the Earth on its axis presents us with an obvious means of defining a coordinate system for the surface of the Earth. The two points where the rotation axis meets the surface of the Earth are known as the *north pole* and the *south pole* and the great circle perpendicular to the rotation axis and lying half-way between the poles is known as the *equator*. Great circles, which pass through the two poles, are known as *meridians*. Small circles, which lie parallel to the equator, are known as *parallels*. Figure 5 and Figure 6 illustrate these definitions.

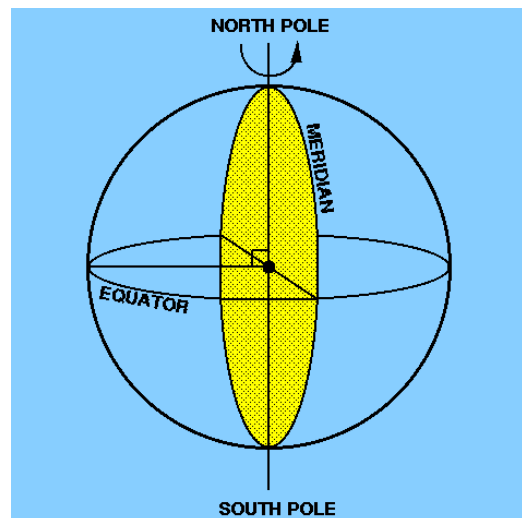
**Figure 5:**

The *north pole*, the *south pole*, the *equator* and a *parallel*



**Figure 6:**

A Meridian



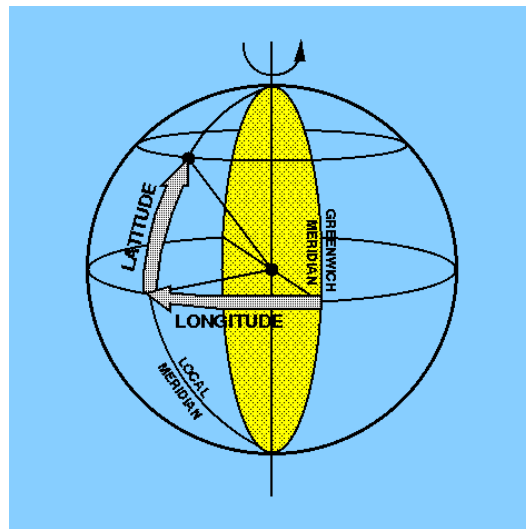
Two coordinates, *latitude* and *longitude*, can be used to define the location of any point on the Earth's surface, as illustrated in Figure 7.

The longitude of a point is measured east or west of the Greenwich Meridian along the equator. The longitude is the angular distance between the local meridian (*which passes through the point*) and the Greenwich meridian (*which passes through the Royal Greenwich Observatory in London*). Because the Earth is rotating, it is possible to express longitude in time units as well as angular units. The Earth rotates by  $360^\circ$  in about 24 hours. Hence, the Earth rotates  $15^\circ$  of longitude in 1 hour,  $1^\circ$  of longitude in 4 minutes,  $1'$  (one minute of longitude) in 4 seconds and  $1''$  (one second of longitude) in  $1/15$  second.

The latitude of a point is the angular distance north or south of the equator, measured along the meridian passing through the point. A related term is the *co-latitude*, which is defined as the angular distance between a point and the closest pole as measured along the meridian passing through the point. In other words, the co-latitude =  $90^\circ - \text{latitude}$ .

**Figure 7:**

Using Latitude and Longitude



Distance on the Earth's surface is usually measured in *nautical miles*, where one nautical mile is defined as the distance subtending an angle of one minute of arc at the Earth's center. A speed of one nautical mile per hour is known as one *knot* and is the unit in which the speed of a boat or an aircraft is usually measured.

Note that because the Earth is not a true sphere (it is actually a *geoid*) the actual measurement of position and distance on the Earth's surface is more complicated than outlined above but the above description is accurate enough for our purposes.

## EXAMPLE PROBLEMS

### 1. The latitude and longitude of

Sheffield are  $53^{\circ}23'12''$  N and  $1^{\circ}28'07''$  W, respectively.

The latitude and longitude of

Sydney are  $33^{\circ}55'24''$  S and  $151^{\circ}17'03''$  E, respectively.

*(Note in these calculation, North Latitude and West Longitude are both positive, while South Latitude and East Longitude are both negative)*

**What is the difference in the latitude of the two cities in decimal degrees?**

$$\text{Difference in latitude} = 53^{\circ}23'12'' - (-33^{\circ}55'24'') = 87^{\circ}18'36''$$

$$\text{Difference in latitude} = 87^{\circ} + (18/60)^{\circ} + (36/3600)^{\circ} = 87.3100^{\circ}$$

**What is the difference in the longitude of the two cities in decimal degrees?**

$$\text{Difference in longitude} = 1^{\circ}28'07'' - (-151^{\circ}17'03'') = 152^{\circ}45'10''$$

$$\text{Difference in longitude} = 152^{\circ} + (45/60)^{\circ} + (10/3600)^{\circ} = 152.7528^{\circ}$$

**What is the difference in longitude in hours, minutes and seconds of time?**

$$\text{Difference in longitude} = 24 \cdot 152.7528 / 360 = 10.18352^{\text{h}}$$

$$\text{Difference in longitude} = 10^{\text{h}} (0.18352 \cdot 60)^{\text{m}} = 10^{\text{h}}11.0112^{\text{m}}$$

$$\text{Difference in longitude} = 10^{\text{h}}11^{\text{m}} (0.0112 \cdot 60)^{\text{s}} = 10^{\text{h}}11^{\text{m}}0.67^{\text{s}}$$

**What are the co-latitudes of Sheffield and Sydney?**

$$\text{Latitude of north pole} = 90^{\circ} \text{ N}$$

$$\text{co-latitude of Sheffield} = 90^{\circ} - 53^{\circ}23'12'' = 36^{\circ}36'48''$$

$$\text{Latitude of south pole} = 90^{\circ} \text{ S}$$

$$\text{co-latitude of Sydney} = 90^{\circ} - 33^{\circ}55'24'' = 56^{\circ}4'36''$$

### 2. Given that the mean radius of the Earth as 6370 km, convert the nautical mile and the knot into miles and mph.

$$\text{Circumference of the Earth} = 2 \pi \cdot 6370 \text{ km}$$

$$\text{Number of arc-minutes in } 360^{\circ} = 360 \cdot 60 = 21600'$$

$$\text{Length of arc subtended by } 1' = 2 \pi \cdot 6370 / 21600 = 1.85 \text{ km}$$

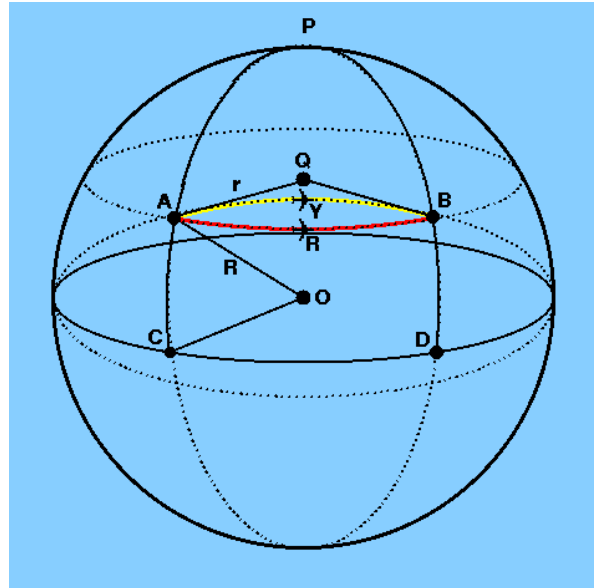
$$\text{Therefore, } 1 \text{ nautical mile} = 1.85 \text{ km} = 1.85/1.61 \text{ miles} = 1.15 \text{ miles}$$

$$\text{and, } 1 \text{ knot} = 1.85 \text{ km/h} = 1.15 \text{ mph.}$$

3. How much longer will it take to fly from Sheffield to Petropavlovsk in Russia along the parallel compared to the great circle route? Assume that Sheffield and Petropavlovsk are at the same latitude ( $53^{\circ}23' \text{ N}$ ), the longitude of Sheffield and Petropavlovsk are  $1^{\circ}28' \text{ W}$  and  $158^{\circ}42' \text{ E}$ , respectively, and the plane is flying at 500 knots.

Figure 8:

Two flight paths from Sheffield to Petropavlovsk in Russia



Let A and B in Figure 8 represent Sheffield and Petropavlovsk, so that the parallel route is denoted by the red arc  $ARB$  and the great circle route is denoted by the yellow arc  $AYB$ . If the meridians  $PAC$  and  $PBD$  are drawn from the north pole  $P$  through  $A$  and  $B$  to the equator  $CD$ , triangle  $PAYBP$  is a spherical triangle. Applying the cosine formula, we may then write

$$\cos AYB = \cos AP \cos BP + \sin AP \sin BP \cos APB ; \text{ where } AP = BP = 90^{\circ} - 53^{\circ}23' = 36^{\circ}37' = 36.6167^{\circ} \text{ and } APB = 1^{\circ}28' - (-158^{\circ}42') = 160^{\circ}10' = 160.1667^{\circ}.$$

Substituting these numbers into the cosine formula gives

$$\cos AYB = (\cos 36.6167^{\circ})^2 + (\sin 36.6167^{\circ})^2 \cos 160.1667^{\circ} ; \text{ thus } AYB = 71.9663 = 71^{\circ}58' = 4318'.$$

The great circle distance between Sheffield and Petropavlovsk is therefore 4318 nautical miles and hence it will take  $4318/500 = 8.636 \text{ h} = 8^{\text{h}}38^{\text{m}}$  to complete the journey via the yellow (upper) arc in Figure 8.

The distance between Sheffield and Petropavlovsk along the parallel of latitude  $53^{\circ}23' \text{ N}$  (a measurement often referred to as the *departure*) can be calculated as follows:

The circumference of the parallel at latitude  $53^{\circ}23' \text{ N} = 2\pi r$ , where  $r = R \cos AOC$ ,  $AOC = 53^{\circ}23' = 53.3833^{\circ}$  and  $R = \text{radius of the Earth} = 3443$  nautical miles.

The red arc  $ARB$  in Figure 8 covers only a fraction of this circumference, where the fraction is given by  $AQB/360^{\circ}$  and  $AQB$  is given by the difference in longitude of  $A$  and  $B$ . So,

$$ARB = (160.1667/360^{\circ}) \cdot 2\pi \cdot 3443 \cdot \cos 53.3833 = 5741 \text{ nautical miles.}$$

Hence it will take  $5741/500 = 11.482 \text{ h} = 11^{\text{h}}29^{\text{m}}$  to complete the journey along the red (lower) arc in Figure 8. Thus, the journey between Sheffield and Petropavlovsk is  $2^{\text{h}}51^{\text{m}}$  quicker along the great circle route than along the parallel.