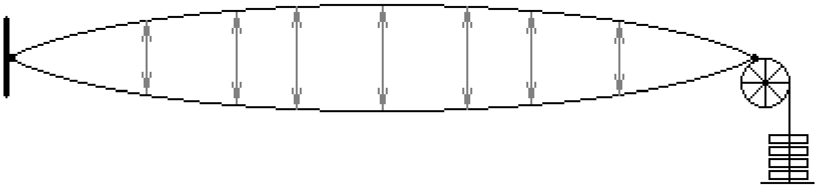
Name: ______________________________________ **Period:** ___ **Due Date:** ____________

LAW of STRINGS

Purpose: To examine the law of strings.

Theory: When a string is stretched under tension, it has a natural vibration frequency. We can investigate this frequency, called the fundamental, by plucking or bowing the string. The fundamental resembles the string in the following diagram.



The weight of the hanging mass at the right provides the tension. The pulley at the right defines one end of the oscillating string while the clamp or wall at the left defines the other end. If we change the tension in the string by changing the amount of weight hanging from the pulley, we change the fundamental frequency. We can also change the fundamental frequency by changing the length of the string between the pulley and wall.

When we pluck a string, the disturbance travels back and forth along the string, but to our eyes the string appears to simply oscillate up and down, as indicated by the gray arrows in the diagram.

The relationship between tension and pitch (*frequency*) is determined by how fast the wave moves on the string.

First, we need to find the speed of the wave. The speed of a wave on a string is related to the tension on the string, the length of the string, and the mass of the string. Specifically, the tension equals:

$$F_T = \frac{mv^2}{L}; v = \sqrt{F_T L / m}; v = \sqrt{F_T / M_L}$$

Where F_T is the tension in the string, m is the mass of the vibrating string, v is the velocity of the wave, L is the length of the vibrating string, and M_L is the mass per unit length of the string, $M_L = m/L$. M_L is assumed uniform.

A second relationship involving the velocity of a wave concerns its frequency and wavelength, and applies to its harmonics, as well:

$$v = f_1 \lambda_1 = f_2 \lambda_2 = f_3 \lambda_3 = f_4 \lambda_4 = f_5 \lambda_5 = ...$$

In other words, all frequencies travel at the same speed. Any increase in the frequency must therefore be matched in the same ratio by a decrease in the wavelength. Likewise, a decrease in the frequency must be matched in the same ratio by an increase in the wavelength. The fundamental frequency is also known as the first harmonic, and the subscripts denote the number of the harmonic, starting with f_1 being the fundamental frequency. The wavelengths of the fundamental and its harmonics are similarly denoted as λ_1, λ_2, λ_3, λ_4…. Some of the higher harmonic frequencies fall on the octaves above the fundamental; some do not

For example, frequency f_1 corresponds to the fundamental, or first harmonic, of the full-length string.

Frequency f_2 corresponds to the first overtone, or second harmonic, and is twice the frequency or one octave above the fundamental. Frequency f_4, corresponds to the third overtone, or the fourth harmonic, and is quadruple the frequency or two octaves above the fundamental. The frequency named f_8, corresponds to the seventh overtone, or the eighth harmonic, and is three octaves above the fundamental. The frequency named f_{16}, corresponds to the fifteenth overtone, or the sixteenth harmonic, and is four octaves above the fundamental.

The other harmonics fall on or very close to other tones in the octaves above the fundamental. If the fundamental corresponds to the tone **C**, then f_2 is **C** in the next higher octave and f_4 is **C** in the octave above that. Frequency f_3, on the other hand, which is 3× the fundamental, corresponds to **G** in the octave above the fundamental. Frequency f_5, which is 5× the fundamental, comes closest to **E** in the second octave above the fundamental. The positions of other harmonics are shown in the table below.

Because the higher harmonics mark off integer multiples of the fundamental frequency, they do not define a musical scale. It is, however, interesting to see how the harmonics are scattered throughout the higher octaves. If we arbitrarily assign the name $\mathbf{C_0}$ to the fundamental, the higher harmonics correspond to the following tones. (*In the modern, equal-tempered music scale the tone named* $\mathbf{C_0}$ *actually has a frequency of 16.35 Hz, so we will use that frequency for this example.*)

Harmonic	Harmonic Frequency (*Hz*)	Equal-Tempered Tone and Frequency (*Hz*)
1st harmonic	**16.35**	$\mathbf{C_0}$ (16.35)
2nd harmonic	**32.70**	$\mathbf{C_1}$ (32.70)
3rd harmonic	**49.05**	$\mathbf{G_1}$ (49.00)
4th harmonic	**65.41**	$\mathbf{C_2}$ (65.41)
5th harmonic	**81.75**	$\mathbf{E_2}$ (82.41)
6th harmonic	**98.11**	$\mathbf{G_2}$ (98.00)
7th harmonic	**114.46**	$\mathbf{A^\#_2}$(116.54)
8th harmonic	**130.81**	$\mathbf{C_3}$ (130.81)
9th harmonic	**147.16**	$\mathbf{D_3}$ (146.83)
10th harmonic	**163.52**	$\mathbf{E_3}$ (164.81)
11th harmonic	**179.87**	$\mathbf{F_3}$ (174.61)
12th harmonic	**196.22**	$\mathbf{G_3}$ (196.00)
13th harmonic	**212.57**	$\mathbf{G^\#_3}$(207.65)
14th harmonic	**228.92**	$\mathbf{A^\#_3}$(233.08)
15th harmonic	**245.27**	$\mathbf{B_3}$ (246.94)
16th harmonic	**261.63**	$\mathbf{C_4}$ (261.63)

This table illustrates the interesting fact that most of the harmonics of any fundamental frequency will fall on or near to other tones in the same scale.

While some of the tones may be a little flat or a little sharp, the differences are all small.

This alignment explains why different instruments sound differently. Each instrument, even instruments of the same basic design by different makers, will produce different admixtures of the higher harmonics. Still, all these instruments produce sounds that are musical because all the overtones are near real tones in the scale.

The variations among instruments merely add to the unique tonal qualities of each instrument, which are often described using words like “color,” “character,” “warmth,” and “texture.”

Some horns use these higher harmonics to play certain tones. This is accomplished with breath control. Most stringed instruments cannot easily be induced to play the higher harmonics on the full-length string. Stringed instruments can play different tones on one string, however, when the musician changes the length of the string thereby changing the wavelength of the fundamental, thereby changing the frequency of the fundamental.

Even though most of us do not “hear” the higher harmonics, we do sense them. Without the upper harmonics all instruments would sound like tuning forks. We don’t really want our instruments to produce pure tones. The richness of our musical heritage comes as much from the quality of the sounds created by the instruments as it does from the creativity of the composers. Without the upper harmonics most music would sound lifeless.

When a string vibrates at its fundamental frequency, $\boldsymbol{f_1}$, the standing wave has a wavelength, λ_1, equal to twice the length of the string, i.e. only half of the wave is on the string at any moment. Each higher harmonic, which has a shorter wavelength than its predecessor, adds an additional one-half of its wavelength onto the string, as follows:

$$L = ({}^{1}\!/2)\,\lambda_1 = ({}^{2}\!/2)\,\lambda_2 = ({}^{3}\!/2)\,\lambda_3 = ({}^{4}\!/2)\,\lambda_4 = ({}^{5}\!/2)\,\lambda_5 = \ldots = ({}^{n}\!/2)\,\lambda_n$$

If we need to calculate the wavelength for any harmonic, we need only solve one of these individual equations for the specific wavelength, as follows:

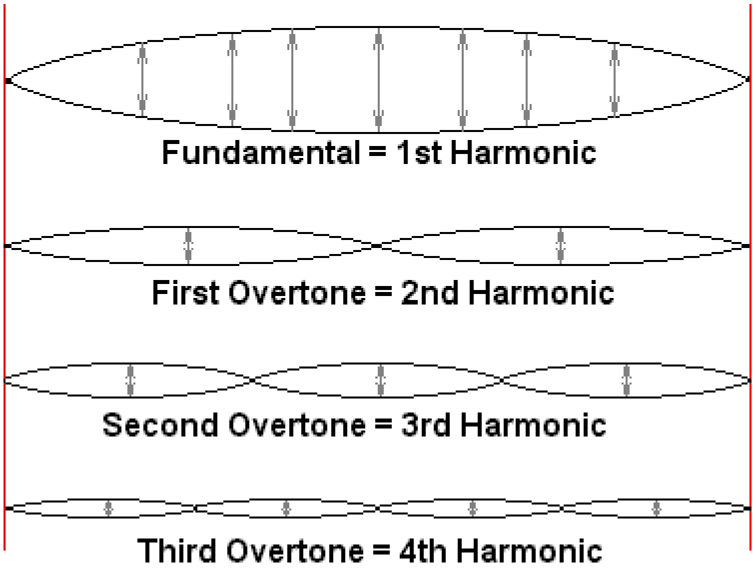
$$\lambda_1 = ({}^{2}\!/{}_{1})L,\;\; \lambda_2 = ({}^{2}\!/{}_{2})L,\;\; \lambda_3 = ({}^{2}\!/{}_{3})L,\;\; \lambda_4 = ({}^{2}\!/{}_{4})L,\;\; \lambda_5 = ({}^{2}\!/{}_{5})L,\; \ldots,\;\; \lambda_n = ({}^{2}\!/{}_{n})L$$

If we wish to focus, as we do in this lab, specifically on the fundamental vibration, we can substitute $\boldsymbol{f_1} \times \lambda_1$ for the velocity and 2**L** for the wavelength in the tension equation to calculate the fundamental frequency:

$$F_T = \frac{m(f_1 \cdot 2L)^2}{L} \quad \text{therefor e} \quad f_1^2 = \left(\frac{1}{4 \cdot L \cdot m}\right) F_T$$

The second equation above demonstrates a linear relationship between the tension on a string and the square of its fundamental frequency of vibration, i.e., for a fixed cavity length and constant string mass, $\boldsymbol{f_1^2} \propto \mathbf{F_T}$.

The Higher Harmonics



If we use a mechanical oscillator to control the string motion, rather than simply plucking it, we can force it to exhibit the higher harmonics one at a time. The first few of these higher harmonics are shown in the figure on the right. Here the second, third, and fourth harmonics are drawn along with the fundamental.

The upper harmonics and the fundamental all occur on the same string, placed under the same tension, and having the same length, and same mass per unit length. Therefore, the velocity of the wave is the same for all harmonics. The string is capable of producing all these harmonics at the same time. When plucked, if there is enough tension in the string, small amounts of these upper harmonics can be detected along with the fundamental. The higher harmonics are usually muted relative to the fundamental. That is why we have to use the mechanical oscillator to see them singly.

Thus, we can double the frequency and halve the original wavelength; or we can triple the frequency and cut the wavelength to one-third of the original; or we can increase the frequency by a factor of four and decrease the wavelength to one-fourth of the original. The string does not do this efficiently by itself with simple plucking, although there will be some of the character of these harmonics in any tone. To study them we need to use the mechanical oscillator to tune into these higher frequencies on the full-length string.

When a musician plays a stringed instrument, these higher harmonics, and other tones, are reached by shortening the string. By pressing the string against a fret or the fingerboard the musician effectively creates a shorter string. The shorter string might correspond to an upper harmonic of the fundamental or it might corresponding to the length for another tone. Plucking or bowing the shortened string is the musician's way of reaching these higher frequencies. Harmonics of other frequencies can also be obtained this way by creating nodes for other tones, not just for the fundamental frequency of that particular string.

When we talk about oscillating strings, it is inevitable that we will have to talk about instruments and music. Here are two facts that you need to keep in mind any time we talk about music.

First, an ***octave*** is a doubling of the frequency. Two octaves are a double doubling ($2\times2 = 2^2 =$ *4 times*), a quadrupling, of the frequency. Three octaves is a triple doubling ($2\times2\times2 = 2^3 =$ *8 times*) of the frequency.

There are 13 tones in an octave. The tones of one octave are: C, C#, D, D#, E, F, F#, G, G#, A, A#, B, C.

Note that there are no sharps or flats between B and C or between E and F.

On the modern, equal-tempered chromatic scale the ratio of the frequencies of any two consecutive tones in this scale is a constant. This uniformity of these ratios, called intervals, has not always been the case. Classical tuning used integer ratios to define the most desirable ratios of frequencies between the first tone in an octave and the higher-frequency members of the same octave. The frequencies of the tones in the equal tempered scale come close but are not exactly equal to those classical frequencies, as we shall see.

One feature of music that is seldom mentioned, but cannot be ignored, is that our outer and inner ears are also instruments. They have response characteristics of their own. It is a subtle and interesting question whether or not a perceived overtone in a musical chord played on given instruments is due to the characteristics of the instruments playing the chord or to the characteristics of the human ear hearing it. If strings can produce tones at higher harmonics, it is also possible that when we hear the fundamental our ears respond by sending a signal corresponding to phantom harmonics to our brains. Such a phantom tone would be indistinguishable from a real tone unless careful and difficult measurements could be made to prove the reality of the overtone.

The other fact to remember is that the harmonics, which are produced by the instruments, might not be heard by our audio perception or we might subtly alter the perceived relative intensities of the various overtones thus changing the perception of these sounds in ways that no measuring instrument can detect.

Note on calculating M_L, the mass per unit length of the string or wire in Part One.

We know that velocity is equal to frequency × wavelength. We also know the equation for the speed on a string or wire in terms of the tension and the mass per unit length. Set the two terms, which are each equal to the same velocity, equal to each other. Then set the wavelength of the fundamental equal to twice the length of the string or wire. After rearrangement, the following equation for mass per unit length, $\mathbf{M_L}$, is obtained.

$$v = f_1\lambda_1 \text{ , and } v = \sqrt{F_T/M_L} \text{ , therefore } f_1\lambda_1 = \sqrt{F_T/M_L} \text{ , and } M_L = F_T/f_1^2\lambda_1^2$$

We know that the fundamental wavelength equals twice the string or wire length, so substitute once more to get

$$M_L = F_T/4f_1^2L^2$$

We can use this equation to calculate $\mathbf{M_L}$ any time we know $\mathbf{F_T}$, $\boldsymbol{f}_1$ and $\mathbf{L}$.

Part One: Relationship between Tension and Frequency (*pitch*). Tuning a string/wire instrument.

(A) Procedure - string:

Determine the frequency of the fundamental vibration of the string in our demonstration. Unfortunately, at low frequencies the fundamental usually presents itself over a wide range of frequencies generated by the mechanical oscillator. Fortunately, the second and third harmonic frequencies present over a narrower ranges of frequencies from the mechanical oscillator. Therefore, our procedure is to find one of these higher harmonics and divide by the appropriate integer, equal to the number of anti-nodes, to calculate the fundamental frequency with greater precision and accuracy.

In other words, if you find the second harmonic, with one node and two anti-nodes, divide that frequency by two to determine the fundamental. If you find the third harmonic, with two nodes and three anti-nodes, divide that frequency by three to find the fundamental frequency. Do not go higher than the third harmonic as the higher frequencies tend to produce more broken strings.

Add tension to the string by adding more mass to the hanger, then find and record the new fundamental frequency. Start with the smallest mass of about 300 g = 0.300 kg, and add 100 to 200 grams at each step. Be careful not to overload the string. 1200 g = 1.200 kg should be a maximum.

Data Table A

Mass of the string hanger: $\mathbf{m}_{\text{hanger}}$ = _____________ kg; Length of the oscillating string: **L** = ___________ m

$\mathbf{m}_{\text{added to hanger}}$ *(kg)*	$\mathbf{m}_{\text{hanging, Total}}$ ($\mathbf{m}_{\text{hanger}}$ + $\mathbf{m}_{\text{added to hanger}}$) *(kg)*	$\mathbf{w}_{\text{hanging, Total}}$ ($\mathbf{m}_{\text{hanging, Total}}\,\mathbf{g}$) *(N)*	**Tension in String** *(N)*	f_1, **Fundamental** *(Hz)*	$\mathbf{M}_{\text{L}}$ (*mass per unit length*) (*kg/m*)
_________	_________	_________	_________	_________	_________
_________	_________	_________	_________	_________	_________
_________	_________	_________	_________	_________	_________
_________	_________	_________	_________	_________	_________
_________	_________	_________	_________	_________	_________
_________	_________	_________	_________	_________	_________
_________	_________	_________	_________	_________	_________
_________	_________	_________	_________	_________	_________

Average mass per unit length = $\mathbf{M}_{\textbf{L-average-string}}$ = _________

Quick Conclusion:
As you add mass to the hanger and increase the tension in the string, how does the fundamental frequency of the string change? It…

[***increases*** / ***decreases***].

Part One-continued: Relationship between Tension and Frequency (*pitch*)

(B) Procedure - wire: Determine the frequency of the fundamental vibration of the wire in our sonometer. Unfortunately, at low frequencies the fundamental usually presents itself over a wide range of frequencies generated by the mechanical oscillator. Fortunately, the second and third harmonic frequencies present over narrower ranges of frequencies from the mechanical oscillator. Therefore, our procedure is to find one of these higher harmonics and divide by the appropriate integer, equal to the number of anti-nodes, to calculate the fundamental frequency with greater precision and accuracy.

In other words, if you find the second harmonic, with one node and two anti-nodes, divide that frequency by two to determine the fundamental. If you find the third harmonic, with two nodes and three anti-nodes, divide that frequency by three to find the fundamental frequency. Do not go higher than the third harmonic as the higher frequencies tend to produce more broken wires.

Add tension to the wire by adding more mass to the hanger, then find and record the new fundamental frequency. Start with the smallest mass of about 400 g = 0.400 kg, and add 100 to 200 grams at each step. Be careful not to overload the wire. 1500 g = 1.500 kg should be maximum.

Data Table B

Mass of the wire hanger: $\mathbf{m}_{\text{hanger}}$ = ______________ kg; Length of the oscillating wire: **L** = ____________ m

$\mathbf{m}_{\text{added to hanger}}$ *(kg)*	$\mathbf{m}_{\text{hanging, Total}}$ ($\mathbf{m}_{\text{hanger}}$ + $\mathbf{m}_{\text{added to hanger}}$) *(kg)*	$\mathbf{w}_{\text{hanging, Total}}$ ($\mathbf{m}_{\text{hanging, Total}}$ **g**) *(N)*	**Tension in Wire** *(N)*	f_1, **Fundamental** *(Hz)*	$\mathbf{M}_L$ (*mass per unit length*) (*kg/m*)

Average mass per unit length = $\mathbf{M}_{\text{L-average-wire}}$ = __________

Quick Conclusion:
As you add mass to the hanger and increase the tension in the wire, how does the fundamental frequency of the wire change? It…

[***increases*** / ***decreases***].

Part One: Analysis: string

Graph Frequency squared (*vertical axis*) vs. Tension (*horizontal axis*).
Graph the string data on a separate graph created in its own separate graphing file. Use the proportional function, y = Ax, to fit the data on the graph. This should be a straight line through he origin. Each of you must create your own graph. Include your name, Graph I, and Lab 23 in the text box with your graph.

Call this Graph I.
(*Suggested data columns for the data table in this file are*: $\mathbf{m}_{\text{hanger}}$, $\mathbf{m}_{\text{added to hanger}}$, $\mathbf{m}_{\text{hanging, Total}}$, $\mathbf{w}_{\text{hanging, Total}}$, $\mathbf{F_T}$, f_{string}, and f_{string}^2. *The columns for* $\mathbf{m}_{\text{hanger}}$, $\mathbf{m}_{\text{added to hanger}}$ *and the string frequency,* f_{string}, *need to be entered by hand. Columns* $\mathbf{m}_{\text{hanging, Total}}$, $\mathbf{w}_{\text{hanging, Total}}$, $\mathbf{F_T}$, *and* f_{string}^2 *can all be created as calculated columns.*)

Note on the line below the slope of Graph I.

String: Slope: ___________________ $\mathrm{Hz^2}/\mathrm{N}$

Significance of the slope: The last equation on page 3 shows that the slope of this graph = $1/(4\cdot\mathbf{L}\cdot\mathbf{m}_{\text{string}})$. Therefore, the mass of the oscillating string, $\mathbf{m}_{\text{string}}$, is given by the equation; $\mathbf{m}_{\text{string}} = 1/(4\cdot\mathbf{L}\cdot\mathbf{slope})$, where **L** is the length of the string.

Length of oscillating string, **L** = _____________ m (*from Data Table A*)

Mass of the oscillating String = $\mathbf{m}_{\text{string}}$ = __________________ kg (*calculated using the slope of Graph I*)

Now that you know the mass and length of the oscillating string, calculate mass per unit length, $\mathbf{M_L}$, in kg/m.

Mass of the string per unit length = $\mathbf{M}_{\text{L-string}} = \mathrm{m}_{\text{string}} / \mathrm{L}$ = ________________ kg/m

Compare the average mass per unit length obtained in Data Table A with the mass per unit length obtained using the mass obtained from Graph I and dividing by the string length.

Are the two mass-per-unit-length determinations for the **string** in reasonably good agreement? **[yes / no]**

Calculate the percent difference between the two mass-per-unit length values for the string using the value calculated using the mass from Graph I as your reference value.

String: %Difference in $\mathrm{M_L} = 100\%\times|\mathbf{M}_{\text{L-average-string}} - \mathbf{M}_{\text{L-string}}| / \mathbf{M}_{\text{L-string}}$ = ____________________ %

In our straight-line graph of frequency-squared vs tension, the slope has units of $\mathrm{Hz^2/N}$, according to the analysis above. Remember that Hz = 1/s. Show that the units on both sides of the equation are equal and that the slope really has units of $\mathrm{Hz^2/N}$.

Write just the units for each term in the equation: f^2 (________) = slope (________) × $\mathbf{T_F}$ (________)

Reduce the slope to its most basic units (in terms of kg, m, s): $\mathrm{Hz^2}/\mathrm{N}$ = ______________________________

Part One: Analysis: wire **Graph Frequency squared (*vertical axis*) vs. Tension (*horizontal axis*).**

Graph the wire data on a separate graph created in its own separate graphing file. Use the proportional function, y = Ax, to fit the data on the graph. This should be a straight line through he origin. Each of you must create your own graph. Include your name, Graph II, and Lab 23 in the text box with your graph.

Call this Graph II.

(*Suggested data columns for the data table in this file are:* $\mathbf{m}_{\text{hanger}}$, $\mathbf{m}_{\text{added to hanger}}$, $\mathbf{m}_{\text{hanging, Total}}$, $\mathbf{w}_{\text{hanging, Total}}$, $\mathbf{F_T}$, f_{wire}, and f_{wire}^2,. *The columns for* $\mathbf{m}_{\text{hanger}}$ *and* $\mathbf{m}_{\text{added to hanger}}$ *and the wire frequency,* f_{wire}, *need to be entered by hand. Columns* $\mathbf{m}_{\text{hanging, Total}}$, $\mathbf{w}_{\text{hanging, Total}}$, $\mathbf{F_T}$, and f_{wire}^2 *can all be created as calculated columns.*)

Note on the line below the slope of Graph II.

Wire: Slope: ___________________ Hz^2/N

Significance of the slope: The last equation on page 3 shows that the slope of this graph = $1/(4\cdot L\cdot m_{\text{wire}})$. Therefore, the mass of the oscillating wire, $\mathbf{m}_{\text{wire}}$, is given by the equation; $\mathbf{m}_{\text{wire}} = 1/(4\cdot L\cdot \text{slope})$, where **L** is the length of the wire.

Length of oscillating wire, **L** = _____________ m (*from Data Table B*)

Mass of the oscillating wire = $\mathbf{m}_{\text{wire}}$ = ___________________ kg (*calculated using the slope of Graph II*)

Now that you know the mass and length of the oscillating wire, calculate mass per unit length, $\mathbf{M_L}$, in kg/m.

Mass of the wire per unit length = $\mathbf{M}_{\text{L-wire}} = m_{\text{wire}}/L$ = ________________ kg/m

Compare the average mass per unit length obtained in Data Table B with the mass per unit length obtained using the mass from Graph II and dividing by the wire length.

Are the two mass-per-unit-length determinations for the **wire** in reasonably good agreement? **[yes / no]**

Calculate the percent difference between the two mass-per-unit-length values for the wire using the value calculated using the mass from Graph II as your reference value.

Wire: %Difference in $\mathbf{M_L} = 100\%\times|\mathbf{M}_{\text{L-average-wire}} - \mathbf{M}_{\text{L-wire}}| / \mathbf{M}_{\text{L-wire}}$ = ____________________ %

Part Two: The Relationship between Length and Frequency (*Pitch*). Playing additional tones on a single string involves using frets or a fingerboard to shorten the string and thereby create new frequencies on the shortened string. In this section we will use the string, but not the wire, but the wire works just as well and would give the same general, but not same numerical, results.

Data Table C

$\mathbf{m}_{\text{hanger}}$ = _______________ kg $\mathbf{m}_{\text{added to hanger}}$ = _______________ kg

$\mathbf{m}_{\text{hanging, Total}}$ = _______________ kg

$\mathbf{w}_{\text{hanging, Total}}$ = _______________ N

Tension on the string = $\mathbf{T_F}$ = _______________ N

Use a moveable stop to shorten the length of the string. Determine the fundamental frequency by using the mechanical oscillator to initiate the second (*or third*) harmonic frequency and divide by two (*or three*) to obtain the fundamental frequency. Be sure you divide by two (*or three*) because the equation we will use in the analysis was derived assuming we would be looking at the fundamental.

Length of String = **L** *(m)*	Fundamental Frequency = f_1 (*Hz*)	Speed of the wave on the string, $v = f_1\lambda_1 = 2f_1\mathbf{L}$ (*m/s*)
___________	___________	___________
___________	___________	___________
___________	___________	___________
___________	___________	___________
___________	___________	___________
___________	___________	___________
___________	___________	___________
___________	___________	___________
___________	___________	___________
___________	___________	___________
___________	___________	___________

Average speed on the string = v_{average} ___________ m/s

Quick Conclusion: As the length of the string is shortened, the frequency [***increases*** / ***decreases***].

Part Two: Analysis:

Graph your results as Frequency (*vertical axis*) vs. Length (*horizontal axis*).
Graph the data on a separate graph created in its own separate graphing file. Fit your data to the inverse function ($y = A/x$; frequency = $A/\text{length} = A/L$). Record the value of A. This should be a hyperbola, with vertical and horizontal asymptotes on the axes. Each of you must create your own graph. Include your name, Graph III, and Lab 23 in the text box with your graph.

Call this Graph III.
You only need two columns in the data table; one for the length of the string and one for the fundamental frequency. These should both be in manual columns. Rename the two columns that appear in each new graph page to reflect what is actually in each column.

A = ______________

Significance of A: From the equations on page 1, we reach the following conclusion about the fundamental frequency of a string under tension.

We know that $f_1\lambda = v$ thus $f_1(2\mathbf{L}) = v$ and therefore $f_1 = v/(2\mathbf{L}) = (v/2)/\mathbf{L}$

We fit the data to the equation $f_1 = A/\mathbf{L}$ **. Comparison with the equation above shows that A = *v*/2**

This means that A equals one-half the velocity of the wave on the string. From this we conclude that the velocity of the wave on our string is:

v_{graph} = velocity of the wave on our string = 2×A = ____________________ m/s.

Now compare the average speed on the string from **Data Table C** with the speed on the string obtained from **Graph III**.

Are the two speeds in reasonably good agreement? Circle one answer. **[yes / no]**

Calculate the percent difference between the two velocity estimates using the speed obtained from the graph as your reference value.

$\%\text{Difference} = 100\% \times |v_{\text{graph}} - v_{\text{average}}| / v_{\text{graph}}$ = ____________________ %

We can calculate the velocity of the wave on our string yet another way. We already determined the mass per unit length of our string, $\mathbf{M}_{\text{L-string}}$, in Part One. In Data Table C we have already calculated the hanging weight, so we know the tension in the string. Use the equation on page 1 to calculate the velocity directly. Calculate the speed of the wave on the string using information from both Parts One and Two;

$v_{I\&II} = [\mathbf{F}_T / \mathbf{M}_{\text{L-string}}]^{1/2}$ = ____________________ m/s

Questions:

1. What happens to the frequency when the string length is halved? What is the relationship between these two frequencies called?

2. How do you play a scale on a stringed instrument?

3. On an instrument without a string, flute, French horn, or trombone, for example, what oscillates to make the sound? How is the frequency (*pitch*) changed in these instruments?

4. How do you tune a stringed instrument? (*Remember that all stringed instruments have a fixed string length for the fundamental tone of each string. Once that fundamental tone is properly tuned, the string will work for all the other tones that can be played on that string.*)

____

____

____

5. Predict the shapes of the following graphs assuming that all other relevant quantities are held constant: (*frequency and wavelength refer to the frequency and wavelength of the fundamental.*)

   a) Frequency **vs** string length ____________________

   b) Frequency **vs** tension in the string ____________________

   c) Frequency **vs** wavelength ____________________

   d) Velocity of the wave **vs** tension in the string ____________________

   e) Wavelength **vs** string length ____________________

Part Three - Mini-Lesson on Frequency and Scales

Violins, and similar instruments, have no frets on them, thus the string can be adjusted to any length and played at any frequency within its range. A violin player must therefore learn to press with the fingers at exactly the right spot to tune each tone the instrument plays to the other instruments. When several violins play a chord, they will carefully adjust the frequencies (*string lengths*) until they are playing frequencies with intervals equal to whole-number ratios. These are the tones that sound most harmonious to human ears; at least in some cultures.

For example, if one violin plays 400Hz, another 500Hz, another 600Hz and another 800Hz, then they are playing a 'major' chord. Measured from the 400Hz violin, these tones have ratios of

400:400=1:1, 500:400=5:4, 600:400=3:2, and **800:400=2:1**.

If they play 400Hz, 480Hz, 600Hz, and 800Hz, they would be playing a 'minor' chord, with ratios of

400:400=1:1, 480:400=6:5, 600:400=3:2, and **800:400=2:1**.

Instruments that can play any frequency (*violins, violas, cellos, human voices, trombones, etc.*) can tune their chords exactly to these harmonious ratios.

A scale is a set of increasing frequencies that ends with a tone at twice the frequency of the first tone (*that is one octave higher than the first tone*). The Western (*8-tone*) scale uses eight tones, in seven intervals, to reach the octave. There are also half-tones, or semi-tones, between some of the tones, giving a total of thirteen tones and twelve intervals to complete each octave.

The one constant across most of the musical scales around the world and throughout is the octave. How the octave is divided into tones, and where the tones are anchored, distinguish one culture's musical tradition from the others.

<u>The Classic Scale and the Classic Intervals</u>. The Pythagoreans are often given credit for discovering the relationship between frequencies in a ratio of small whole numbers and harmony. The frequency ratios, relative to the first tone, which they found, and which has come down to us, are these:

1:1 9:8 5:4 4:3 3:2 5:3 15:8 2:1

Clearly, not every possible ratio of whole numbers is represented. The ratios they chose produced the most pleasant tones and harmonies to their ears. These ratios are interesting in many ways. For example, if we look at the interval between successive tones we get the following picture:

Classic Scale	Do	⇔	Re	⇔	Mi	⇔	Fa	⇔	Sol	⇔	La	⇔	Ti	⇔	Do
Musical Notation	C_4		D_4		E_4		F_4		G_4		A_4		B_4		C_5
Classic Ratios	1:1		9:8		5:4		4:3		3:2		5:3		15:8		2:1
Classic Intervals		9/8		10/9		16/15		9/8		10/9		9/8		16/15	
Classic Frequencies	264		297		330		352		396		**440**		495		528

There are seven intervals in this scale for each octave. The intervals come in three different sizes;

9:8, 10:9, and 16:15.

The larger intervals (9:8, 10:9) are further divided into semi-tones. The smallest interval (16:15) is too small to subdivide. There is no standardized way to divide the large intervals to create the semi-tones, however.

In the scale where the Do is a C, there are no semi-tones between E and F or between B and C. This Classic scale provides a very harmonious musical scale. It is well suited to instruments that can be tuned as they are played; violin, lyre, cello, etc. It is not as well suited to instruments that must be tuned before they are played; piano being an important example. Changing key for music written for those instruments can be very challenging. The different sized intervals force the composer to make some hard choices when changing keys.

Well-Tempered Scale and Well-Tempered Intervals. It was recognized that what composers and musicians needed was a scale that, while still harmonious, satisfied the needs of an entire orchestra and the musical community at large.

This recognition led to the development of the so-called diatonic scale; a scale with only two different intervals. It also included specific intervals for the semi-tones and came close to producing a 12-tone scale. The new frequencies were not very different from the classical frequencies so that the older musical compositions did not need to be rewritten. The sought after diatonic scale became known as the well-tempered scale.

The well-tempered scale was intended to define all intervals using only two intervals (*diatonic*):

Well-tempered Scale	Do	⇔	Re	⇔	Mi	⇔	Fa	⇔	Sol	⇔	La	⇔	Ti	⇔	Do
Musical Notation	C_4		D_4		E_4		F_4		G_4		A_4		B_4		C_5
Well-tempered Ratios	1:1		9:8		$3^4:2^6$		4:3		3:2		$3^3:2^4$		$3^5:2^7$		2:1
Well-tempered Ratios	1:1		9:8		81:64		4:3		3:2		27:16		243:128		2:1
Well-tempered Intervals		9/8		9/8		256/243		9/8		9/8		9/8		256/243	
Well-tempered Intervals		$3^2/2^3$		$3^2/2^3$		$2^8/3^5$		$3^2/2^3$		$3^2/2^3$		$3^2/2^3$		$2^8/3^5$	
Well-tempered	260.74		293.3		330		347.6		391.1		**440**		495		521.5

Starting with the first tone, which is in the ratio of 1:1 with itself, the ratios of the frequencies of the other tones to the first tone are close to the classic ratios, as can be seen here. Concentrate specifically on the ratios that differ with the classic ratios:

Ratio Comparison	Do	⇔	Re	⇔	Mi	⇔	Fa	⇔	Sol	⇔	La	⇔	Ti	⇔	Do
Musical Notation	C_4		D_4		E_4		F_4		G_4		A_4		B_4		C_5
Classic Ratios	1:1		9:8		5:4		4:3		3:2		5:3		15:8		2:1
(*decimal*)					1.25						1.666		1.875		
Well-tempered Ratios	1:1		9:8		$3^4:2^6$		4:3		3:2		$3^3:2^4$		$3^5:2^7$		2:1
(*decimal*)					1.265						1.6875		1.8984		

The well-tempered ratios are a good approximation of the classical ratios, and all but three are identical.

Notice, too, that every large interval is now $3^2/2^3 = 9/8$ and every small interval is now $2^8/3^5 = 256/243$. The interval ratios of the well-tempered scale, when multiplied together, produce an octave, as required.

$$\{9/8 \times 9/8 \times 256/243 \times 9/8 \times 9/8 \times 9/8 \times 256/243\} = 2 = (\textit{double the frequency}) = 1 \text{ octave.}$$

What's awkward about these intervals, an awkwardness the well-tempered scale shares with the classical scale, is lack of mathematical simplicity associated with picking the semi-tone frequencies. There is no mathematically obvious way to divide the larger intervals. In practice, the semi-tones are chosen to sit 256/243 above the previous tone or 256/243 below the next higher tone. Unfortunately, those two semi-tone options do not have exactly the same frequency, so there is room for some artistic maneuvering, and, for example $C^{\#}$ is not exactly the same tone as D^{b}. Some instruments play it one way, some instruments play it the other way.

For the tones, the well-tempered frequencies are in good agreement with the classical frequencies. We assume that A_4 is set to concert pitch at 440 Hz. This frequency comparison shows that the well-tempered scale is a little flat. On average it is lower in frequency then the classical scale. Here are the frequencies for comparison:

Frequency Comparison Musical Notation	Do ⇔ C_4	Re ⇔ D_4	Mi ⇔ E_4	Fa ⇔ F_4	Sol ⇔ G_4	La ⇔ A_4	Ti ⇔ B_4	Do C_5
Classic Frequencies	264	297	330	352	396	**440**	495	528
Well-tempered Frequencies	260.74	293.33	330	347.65	391.11	**440**	495	521.48

<u>The Equal-Tempered Scale and Equal-Tempered Intervals</u>. By the early nineteenth century, European musicians wanted to devise a new scale where changing the scale up or down by a fixed number of tones resulted in the same tune in a different key. Such a scale is impossible to make using the harmonic intervals in the classic or well-tempered scales. Those scales also make it impossible for musicians and composers to change the key on the fly; while playing. That limits the creative potential of certain musical forms.

The solution, called the equal-tempered scale, used frequencies for which the interval between any two consecutive notes is $2^{1/12}$. The large interval between notes is $(2^{1/12})^2 = 2^{1/6}$. As before, there are five large steps and two small steps in each octave, which starts on a C. The advantage of this method of tuning is that, by using the sharps$^\#$ and flats^b, a scale on a keyboard instrument can start on any tone. The disadvantage is that no tones except the octave tones are perfectly harmonic (*that is to say they do not sound like they are in tune*).

Today, keyboard instruments are usually tuned to this 'equally tempered' scale. Here are the ratios, intervals and standard frequencies for all three scales, beginning with middle C_4 and rising one octave to C_5.

Classic Scale	C_4		D_4		E_4		F_4		G_4		A_4		B_4		C_5
Classic Ratios	1:1		9:8		5:4		4:3		3:2		5:3		5:3		2:1
Classic Intervals		9/8,		10/9,		16/15,		9/8,		10/9,		9/8,		16/15,	
Classic Frequencies	264		297		330		352		396		**440**		495		528

Well-tempered Scale	C_4		D_4		E_4		F_4		G_4		A_4		B_4		C_5
Well-tempered Ratios	1:1		9:8		$3^4:2^6$		4:3		3:2		$3^3:2^4$		$3^5:2^7$		2:1
Well-tempered Intervals		9/8,		9/8,		$2^8/3^5$,		9/8,		9/8,		9/8,		$2^8/3^5$	
Well-tempered Frequencies	260.74		293.33		330		347.65		391.11		**440**		495		521.48

Equal-tempered Scale	C_4		D_4		E_4		F_4		G_4		A_4		B_4		C_5
Equal-tempered Ratios	$2^{0/12}$		$2^{2/12}$		$2^{4/12}$		$2^{5/12}$		$2^{7/12}$		$2^{9/12}$		$2^{11/12}$		$2^{12/12}$
Equal-tempered Intervals		$2^{2/12}$,		$2^{2/12}$,		$2^{1/12}$,		$2^{2/12}$,		$2^{2/12}$,		$2^{2/12}$,		$2^{1/12}$,	
Equal-tempered Frequencies	261.63		293.66		329.63		349.23		392		**440**		493.88		523.25

The American and International standards for concert pitch with the equal-tempered scale differ. The American standard defines the A_4 above middle C_4 to be at exactly 440 Hz (*in the international standard it is 435 Hz*). The western musical scale is historically divided into seven major intervals. With the advent of the equal-tempered scale, it is probably simpler to think of it as having 12 half-intervals. Each half-interval is a step in frequency equal to $2^{1/12}$ above the previous frequency or $2^{-1/12}$ below the following frequency.

Even thought the steps are all in the same ratio, standard musical naming conventions are still used. No sharp$^\#$ or flat^b tones occur between B and C or between E and F, as in the classical scale. Since the sharps$^\#$ and flats^b occur at the same frequency whether calculated from the lower tone or the higher tone, there is only one semi-tone between each tone. What you call it is irrelevant; $C^\#$ has exactly the same frequency as D^b.

Assignment: Try calculating the frequencies of the classic, well-tempered, and equal-tempered scales. Calculate the frequencies of all tones and semi-tones. The frequency differences are not constant. The intervals are constant. This is because our senses operate on a logarithmic scale; we sense (*our auditory equipment hears*) the ratio of the frequencies not the difference between the frequencies.

The ratios of the tones in the classic C-major scale are given below: (*A_4 is a major sixth above middle C_4.*)

Ratio Comparison	C=Tonic	D=second	E=major third	F=fourth	G=fifth	A=major sixth	B=major seventh	C=octave
Classic Ratios	1:1	9:8	5:4	4:3	3:2	5:3	5:8	2:1
Well-tempered Ratios	1:1	9:8	$3^4:2^6$	4:3	3:2	$3^3:2^4$	$3^5:2^7$	2:1
Equal-tempered Ratios	$2^{0/12}=1:1$	$2^{2/12}:1$	$2^{4/12}:1$	$2^{5/12}:1$	$2^{7/12}:1$	$2^{9/12}:1$	$2^{11/12}:1$	$2^{12/12}=2:1$

Note that the small intervals occur between E and F and between B and C. There are no semi-tones (*sharps*$^\#$ *or flats*^{b}) between these pairs of tones. The use of sharps$^\#$ and flats^b (*called accidentals*) should be unnecessary in an equal-tempered musical system. $C_4{}^\#$ has the same frequency as $D_4{}^b$, for example. In the classic and well-tempered scales the sharp$^\#$ and flat^b semitones in the same interval are not exactly equal in frequency. On the classic and well-tempered scales multiply by 256/243 to find the sharp$^\#$ and divide by 256/243 to find the flat^b.

List the frequencies and frequency differences for all the tones of an octave defined on A_4 = 440Hz.

Classic Scale			Well Tempered Scale			Equal-Tempered Scale		
	(*Hz*)	*Δf*		(*Hz*)	*Δf*		(*Hz*)	*Δf*
Do = C_4	_______		Do = C_4	_______		Do = C_4	______	
		_______			_______			_______
$C_4^\#$	_______		$C_4^\#$	_______		$C_4^\#$		
D_4^b	_______		D_4^b	_______		D_4^b	———	
		_______			_______			_______
Re = D_4	_______		Re = D_4	_______		Re = D_4	______	
		_______			_______			_______
$D_4^\#$	_______		$D_4^\#$	_______		$D_4^\#$		
E_4^b	_______		E_4^b	_______		E_4^b	———	
		_______			_______			_______
Mi = E_4	_______		Mi = E_4	_______		Mi = E_4	______	
		_______			_______			_______
Fa = F_4	_______		Fa = F_4	_______		Fa = F_4	______	
		_______			_______			_______
$F_4^\#$	_______		$F_4^\#$	_______		$F_4^\#$		
G_4^b	_______		G_4^b	_______		G_4^b	———	
		_______			_______			_______
Sol= G_4	_______		Sol= G_4	_______		Sol= G_4	______	
		_______			_______			_______
$G_4^\#$	_______		$G_4^\#$	_______		$G_4^\#$		
A_4^b	_______		A_4^b	_______		A_4^b	———	
		_______			_______			_______
La = A_4	__**440**__		La = A_4	__**440**__		La = A_4	__**440**_	
		_______			_______			_______
$A_4^\#$	_______		$A_4^\#$	_______		$A_4^\#$		
B_4^b	_______		B_4^b	_______		B_4^b	———	
		_______			_______			_______
Ti = B_4	_______		Ti = B_4	_______		Ti = B_4	______	
		_______			_______			_______
Do = C_5	_______		Do = C_5	_______		Do = C_5	______	

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