Lesson 00-A0 - Algebra

Algebra is essentially a body of rules and procedures. They tell use how to logically explore relationships between concepts. By manipulating symbols according to these rules and procedures we map out these relationships in a concise and easy to read format. Relationships are described using equations. One concept or combination of concepts equals a second concept or combination. Equations are about more than numbers. They describe relationships. The numbers that describe a particular situation are minor results of the important general relationship described by the general equation.

For example, consider the relationship among the distance traveled (D) by an object moving in a straight line at a given speed (v) for a given time (t). The relationship is

$$D = vt$$

Even such a simple equation contains many details about the relationship. You will learn to read these details for yourself as you gain experience with relationships.

- When we have only one equation, we can find only one unknown quantity. If we know v and t we can solve for D. If we know D and v we can solve for t. If we know D and t we can solve for v. In all cases we need to know two items, called parameters, to find the missing one. That is an inherent property of this relationship.

- This relationship holds only if certain assumptions are valid. The velocity must be constant. The motion must be in a straight line. Time must be measured in uniform increments. Distances must be measured in uniform increments.

- The units of the two known quantities determine the units that must apply to the unknown quantity. If D is measured in meters and t is measured in seconds, then v must be in meters per second.

- The equation tells us that if v = 5 m/s and t = 3 s, then the distance of travel, D, will be 15 meters. But the equation is not limited to just this one case. Remember, it is a general relationship. General relationships apply to lots of cases, as long as all of them satisfy the three requirements above. So all of the following cases are governed by the same equation because they satisfy the same requirements and meet the necessary conditions.

    - ► D = 77 m; v = 11 s; therefore t = 7 s
    - ► D = –120 m; t = 30 s; therefore v = –4 m/s
    - ► and many, many (infinitely many) more

In real life, and in physics class assignments, such problems are often couched in common language. One of your tasks will be to tease out of the language the information needed to determine which relationship, and therefore which equation, must be used to solve the problem contained in the words. The question may be something like one of the following:

- We drove from Fort Worth to Abilene in 63 minutes at 60 miles per hour. How far did we drive?

- A bird took 7 seconds to fly from one telephone pole to the next. The poles are 100 feet apart. Figure out how fast the bird flew between the telephone poles.

In this course you will become very familiar with word problems because most of the physics we study is related to real world activities and actions. Language is our normal mode of communication, but mathematics is our normal mode for solving problems that involve numbers. Translation between the two is therefore an essential skill.

To solve the bird/telephone pole problem there are several rules you must follow. These apply to all the problems and the applicable equations that we study. These are the rules of algebra. Before you start applying algebra to your question, however, it is always a prerequisite that you first decide which relationships are included in the problem and therefore which equation or equations you will use to solve the problem. In this case we've taken care of that step and we already know that the relationship is $D = vt$. Therefore, we can proceed with the algebra.

Logical rules for solving for the unknown quantity in a single equation problem.

1. The same quantity (a constant or a variable) can be added to or subtracted from both sides of the equation at any time without destroying the equality. **Of course, if you decide to add or subtract unequal quantities to the two sides of the equation you will get nonsense as your answer.**

2. At any time both sides of an equation can be multiplied or divided by the same quantity (a constant or a variable) without destroying the equality. **Of course, if you decide to multiply or divide by unequal quantities on the two sides…**

3. Both sides of an equation can be raised to the same power (square, square root, cube, cube root, etc) without destroying the equality. **Of course, if you…**

4. You can take the logarithm to the same base or take the exponential to the same base on both sides of an equation without destroying the equality. **Of course, …**

When manipulating equations using these rules it is always preferable to search for a solution that uses rule 1 before trying rule 2. And it is better to search for a solution that uses rule 2 before trying rule 3. And it is better to search for a solution that uses rule 3 before trying rule 4.

Now, back to the bird problem; the equation that applies is D = vt. Substitute the known quantities into this equation to get 100 ft = v×(7 s). To get the v alone by itself on one side of the equation use the rules of algebra. Rule 1 is no help. Invoking rule 2 we divide both sides of the equation by 7s. This produces a new and equally valid equation; (100 ft) / (7 s) = v×(7 s) / (7 s) = v; hence v = 14.285714 = 14.3 ft/s.

Try another equation. Solve for x in the equation 6x – 11 = 3. Invoking rule 1 we add 11 to both sides of the equation yielding; 6x = 3 + 11 = 14. Then we invoke rule 2 and divide both sides of the equation by 6; hence x = 14/6 = 2⅓.

Try another. Solve for the y in $5y^2 + 17 = 42$. Invoke rule 1 and subtract 17 from both sides of the equation to yield; $5y^2 = 42 - 17 = 25$. Then invoke rule 2 and divide both sides of the equation by 5, to get; $y^2 = 25/5 = 5$. Finally, invoke rule 3 and raise both sides of the equation to the ½ power (take the square root). This produces the following result: $\sqrt{y^2} = y = \pm\sqrt{5} = \pm 2.236068\ldots$. There are two solutions to this equation; +2.236068… and –2.236068….

Try one more. Find the t in $\frac{3}{4}t^2 - 6 = 0$. Invoke rule 1 to move the 6 to the other side of the equation (add 6 to both sides of the equation); $\frac{3}{4}t^2 = 6$. Invoke rule 2 and move the 4 in ¾ to the to the other side of the equation (multiply both sides by 4); $3t^2 = 24$. Invoke rule 2 again to move the 3 to the other side of the equation (divide both sides by 3); $t^2 = 8$. Finally, Invoke rule 3 and take the square root of both sides; $t = \pm\sqrt{8}$. Again, there will be two solutions to the situation described by the quantities involved in this relationship.

Warnings and Trouble Spots

♪ Dividing by fractions can be troublesome. $4/\frac{1}{4}$ is not 1; it is 16. $\frac{1}{4} / 4$ is not 1; it is $^1/_{16}$. The rule to apply here is that a fraction is not changed when the top and bottom are multiplied by the same quantity. This is a variation of rule 2. In the first example, multiply top and bottom by 4 to get 16/1 = 16. In the second example multiply top and bottom by 4 to get $^1/_{16}$.

♪ $(a + b)^2$ is **NOT EQUAL** to $a^2 + b^2$. To multiply binomials, as in this case, or polynomials, write the individual terms out explicitly. Exponentiation is simply a series of multiplications and you should evaluate binomials and polynomials this way. Thus, $(a + b)^2 = (a + b)\bullet(a + b) = aa + ab + ba + bb = a^2 + 2ab + b^2$.

♪ $(^1/_a) + (^1/_b)$ is **NOT EQUAL** to $(^1/_{(a + b)})$. We cannot add two fractions together unless they have the same denominator. We use the rule about fractions again to find a common denominator before we try adding them together. Therefore,

$$(^1/_a)(^b/_b) + (^1/_b)(^a/_a) = (^b/_{ab}) + (^a/_{ba}) = (^b/_{ab}) + (^a/_{ab}) = ^{(a + b)}/_{ab}$$

For example:

$$(^1/_2) + (^2/_5) = (^5/_{10}) + (^4/_{10}) = {}^9/_{10} \quad \textbf{NOT} \ {}^3/_7 \text{ !!!!}$$

♪ Remember that $\sqrt{a}\bullet\sqrt{b} = \sqrt{(ab)}$. $\sqrt{4}\bullet\sqrt{9} = \sqrt{36} = \pm 6$, or $(\pm 2)\bullet(\pm 3) = \pm 6$. And furthermore, $\sqrt{8} = \sqrt{4}\bullet\sqrt{2} = \pm 2\sqrt{2}$. Similar situations come up more frequently than you might think in physics problems; such as $\sqrt{(25gt^2)} = \sqrt{25}\bullet\sqrt{g}\bullet\sqrt{t^2} = \pm 5t\sqrt{g}$.

♪ Given a complex relationship like the following $F_1 = GmM/R^2$, how does F_1 change if m is doubled? The new $F_2 = G(2m)M/R^2 = 2\bullet GmM/R^2 = 2F_1$, that is, F_2 is twice F_1. We say that F is directly proportional to m. Doubling the one, doubles the other.

How does F_1 change if R is doubled? The new $F_3 = GmM/(2R)^2 = \frac{1}{4}\bullet GmM/R^2 = \frac{1}{4}F_1$, which means that F_3 is one-fourth of F_1. We say that F is inversely proportional to the square of R; doubling R decreases F by a factor of 2^2 or 4; tripling R decreases F by a factor 3^2 or 9.