Lesson 00-A3-Logarithms and Exponentials

Logarithms - Imagine that you have identified a certain number, y, that can be expressed as a certain base number, b, raised to a specific exponent, x. We write this in the form of an equation, as

$$y = b^x$$

There are limitations we need to impose on the expression that you must be aware of. If we do not impose these limitations, the logarithm defined in the next section will not be logically consistent. First, b must be a number greater than zero, $b > 0$. Second, b must not be equal to one, $b \neq 1$. Third, as a result of the first requirement, y will always be greater than zero, $y > 0$. On the other hand, x can be any real number, positive or negative, whole, fractional, or irrational number (like; +4, $-^{7}/_{12}$, π or $\sqrt{2}$).

The logarithm is defined as follows: The logarithm, base b, of a positive real number, y, is the exponent, x, of base b that reproduces y. The symbol looks like this

$$\text{If } y = b^x \text{ (for +y, } b > 0 \text{ and } b \neq 1\text{) then, } x = \log_b y$$

In words this reads as, "x is the logarithm to the base b of y." Another way to read it is, "base b raised to the exponent x equals y." The logarithm is an exponent. The hardest part of logarithms for some people is getting used to the concept that exponents, and therefore logarithms, are not always whole numbers.

The logarithm, x in this example, is an exponent. Logarithms behave the way exponents behave. When the numbers they represent are multiplied, the logarithms are added. If the number represented by a logarithm is raised to a power, then the logarithm is multiplied by that power.

Here are simple examples illustrating some of the fundamental rules for logarithms.

- Since $3 = 3^1$; therefore $1 = \log_3 3$ and $1 = \log_n n$ (for all $n > 0$ and $n \neq 1$)

- Since $9 = 3^2$; $2 = \log_3 9$; Since $3 \times 9 = 27$, $1 + 2 = 3 = \log_3 3^3 = \log_3 27$

- Since $(3^2)^2 = 81$; $2 \times 2 = 4 = \log_3 3^4$

- $\log_3 1 = 0$; and $\log_n 1 = 0$, n is a real number such that $n > 0$ and $n \neq 1$.

- Try this on your calculator: 3^0.588975 = 1.909908.

Therefore, by hand we can write; $0.588975 = \log_3 1.909908$

Logarithms will become more important in the second half of the course. By that point we will be interested in only two base numbers. The two bases are 10, the base of the *common logarithms*, and e (= 2.718 281 828 …), the base of the *natural logarithms*. These are distinguished from each other and from all other possible bases using special forms.

For the common logarithms: $10^2 = 100$ therefore $2 = \log 100$. No base is or needs to be indicated when the base is 10

For natural logarithms: $e^2 = 7.389056$ therefore $2 = \ln 7.389056$. The natural logarithms are so important that a special symbol has been universally adopted, i.e. $\log_e$ is always written as ln.

The common logarithm is used in situations where we are using logarithms to simplify mathematics. The natural logarithms are encountered when we study processes where the rate-of-change of a quantity depends on that quantity. This happens a lot.

Logarithms have some cool properties, most of which we've already encountered in this presentation. In general, logarithms obey the following rules

- $\log (ab) = \log a + \log b$; $\ln (ab) = \ln a + \ln b$

- $\log (a/b) = \log a - \log b$; $\ln (a/b) = \ln a - \ln b$

- $\log a^n = n \log a$, $\ln a^n = n \ln a$, *this can be very useful in simplifying equations*

- $\log_b b = 1$; $\log 10 = 1$; $\ln e = 1$

- $\log 1 = 0$; $\ln 1 = 0$

- $10^{\log a} = a$, $e^{\ln a} = a$, *this one may seem confusing until you look at it very carefully and think about it for a while*

The rules above work exactly the same for common logarithms and natural logarithms.

Exponentials – When we use the term exponentials in the course we are referring to exponential expressions of base e raised to a real, positive or negative, exponent. Exponential expressions of this type are the mirror image, if you like, of the natural logarithms. We will not encounter exponentials until late in the 1st semester or early in the 2nd semester.

The relationship between natural logarithms and exponentials can be summarized as follows:

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x$$

To check the equalities in these two expressions take the exponential of both sides of the equation on the left and the natural logarithm of both sides of the equation on the right. This yields

$$e^{\ln e^\wedge x} = e^x \quad \text{and} \quad \ln(e^{\ln x}) = \ln x$$
$$e^x = e^x \quad \text{and} \quad \ln x = \ln x$$

Note also that both expressions equal x, therefore they must equal each other, so

$$\ln e^x = e^{\ln x}$$
$$x \ln e = x$$
$$x = x$$

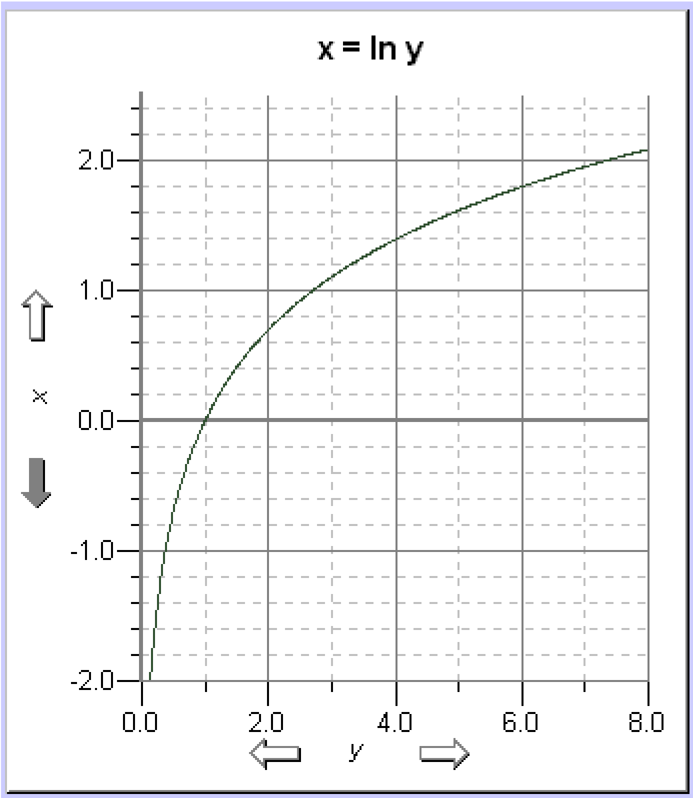
Let's look at the exponential equation, $y = e^x$, just a little more closely.
If $y = 1$, then $x = 0$
If $y = 2$, then $x = \ln(2) = \ln(2^1) = 1 \cdot \ln(2) = 1 \cdot 0.693147\ldots = 0.693147\ldots$
If $y = 4$, then $x = \ln(4) = \ln(2^2) = 2 \cdot \ln(2) = 2 \cdot 0.693147\ldots = 1.386294\ldots$
If $y = 8$, then $x = \ln(8) = \ln(2^3) = 3 \cdot \ln(2) = 3 \cdot 0.693147\ldots = 2.079442\ldots$
If $y = 16$, then $x = \ln(16) = \ln(2^4) = 4 \cdot \ln(2) = 4 \cdot 0.693147\ldots = 2.772589\ldots$

Every time that x increases by 0.693…, y doubles. We say, "as x increases linearly, y increases exponentially." If x is a positive exponent, the value of y increases faster than x increases.

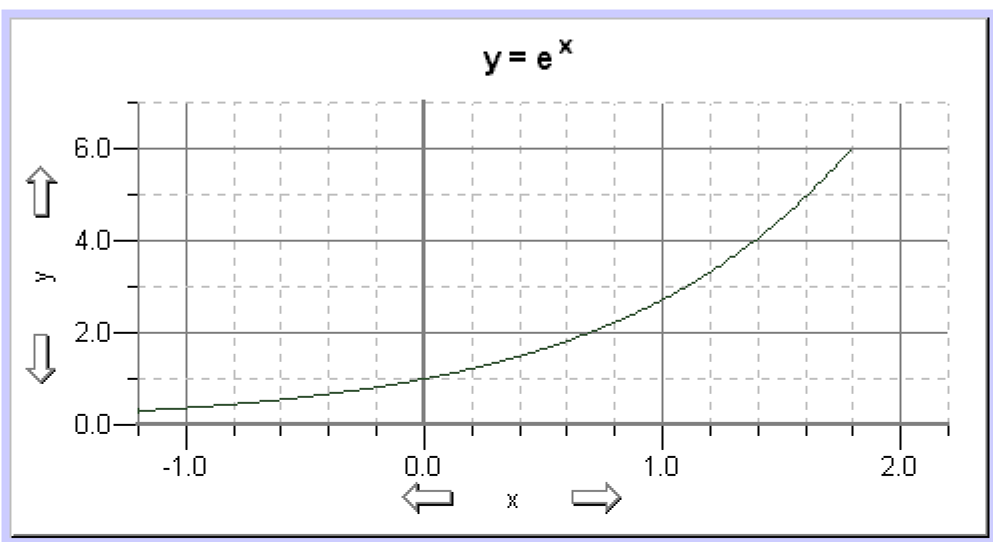
Let's look at the exponential equation, $y = e^{-x}$, just a little more closely.
If $y = 1$, then $x = 0$
If $y = ½$, then $x = \ln(2^{-1}) = -1 \cdot \ln(2) = -1 \cdot 0.693147\ldots = -0.693147\ldots$
If $y = ¼$, then $x = \ln(2^{-2}) = -2 \cdot \ln(2) = -2 \cdot 0.693147\ldots = -1.386294\ldots$
If $y = {}^1/_8$, then $x = \ln(2^{-3}) = -3 \cdot \ln(2) = -3 \cdot 0.693147\ldots = -2.079442\ldots$
If $y = {}^1/_{16}$, then $x = \ln(2^{-4}) = -4 \cdot \ln(2) = -4 \cdot 0.693147\ldots = -2.772589\ldots$

Every time that x decreases by 0.693…, y is cut in half. We say, "as x decreases linearly, y decreases exponentially." If x is a negative exponent, the value of y decreases more slowly than x decreases.

Graphs of the Logarithmic and Exponential Functions



As noted earlier, the logarithmic function is only defined for positive values of y, but x, the logarithm, can be any real number, positive, negative, rational or irrational, even zero.



As noted earlier, the exponential function is defined for all real values of x, but all the possible values of y are positive.