

Lesson 00-A4-Proportionalities and Equations

Proportionalities – Experimentally, we examine effects in nature in pairs. This is an important aspect of experimental design. Even though there may be many variables that effect a given situation, it is important to focus our attention and our experiments on only two at a time. To accomplish that end, it is therefore essential that we keep all the other variables constant while we vary the two of immediate concern.

For example, the gravitational force acting between two bodies, say the Earth and the Moon, is influenced by the two masses and the distance between the objects. That means we have four variables to investigate: mass of the first object, mass of the second object, the distance between them and the force. To study the mutual interaction of force and distance, we need to keep the two masses constant. To study the mutual interaction of force and one mass, we need to keep the distance and the other mass constant. What we hope to learn in these pair-wise studies is the proportionality between two of the variables.

There are four types of proportionality to discuss. Nature most frequently displays these proportionalities, but there are others.

- Direct Proportionality – If we double one of the variables and find that the other one also doubles, we call that a direct proportionality. The two are directly proportional. This is the case for force and mass in gravitational investigations. The force of gravity between two bodies is directly proportional to the mass of the first object. The force of gravity is also directly proportional to the mass of the other object.

We write this $\text{Force} \propto \text{mass}_1$ and $\text{Force} \propto \text{mass}_2$, therefore, $F \propto m_1 \times m_2$

- Inverse Proportionality – If we double one of the variables and find that the other one is cut in half, we call that an inverse proportionality. The two are inversely proportional. This is the case for resistance and cross-sectional area of a wire, as you will learn later in the course. The resistance of the wire is inversely proportional to the cross-sectional area of the wire.

We write this $\text{Resistance} \propto 1 / \text{cross-sectional area}$, $R \propto 1/A$, and $R \propto A^{-1}$

- Direct Proportionality to the Square – If we double one variable and the other increases by $2^2 = 4$, then the first variable is directly proportional to the square of the second variable. The area of a square is directly proportional to the square of its side. The area of the circle is directly proportional to the square of its radius. The surface area of a sphere is directly proportional to the square of its radius.

We write these as $A_{\text{square}} \propto L_{\text{side}}^2$. $A_{\text{circle}} \propto r^2$. $A_{\text{surface of a sphere}} \propto r^2$

- Inverse Proportionality to the Square – If we double one variable and find that the other one is cut to one-fourth of its initial value ($1/2^2 = 1/4$), then we say that the first variable is inversely proportional to the square of the other. If we consider the force of gravity between two bodies once more, we will find that force is inversely proportional to the square of the distance between the bodies.

We write this $F \propto 1/R^2$, where R is the distance between the centers of mass.

It is important to investigate these proportionalities whenever we can find them because they give invaluable information about the relationships among the variables. The first and fourth proportionalities described above, for example, tell us almost everything we need to know in order to work gravitational problems.

What we know about gravity from these proportionalities can be summarized as follows

$$F \propto m_1 m_2 / R^2$$

All that is missing is the proportionality constant. Before worrying about its value, you need to observe something very important about proportionalities. When two experimental variables are proportional, it eliminates whole classes of possible relationships among the variables. They tell us that all the variables must be multiplied together and that there are no additive terms. If F is proportional to m_1 , then there cannot be any term like C.

$$F \text{ cannot be } \propto m_1 m_2 / R^2 + C$$

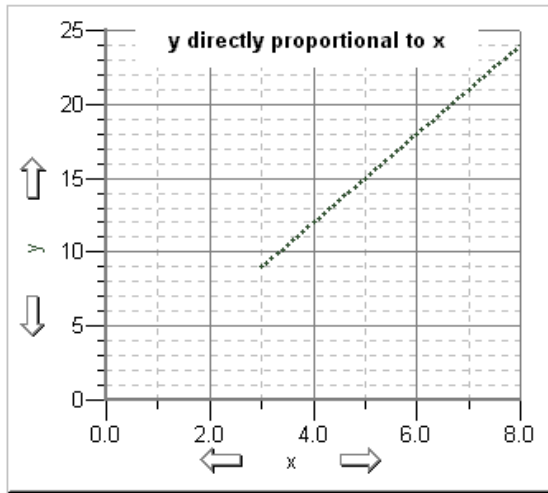
The only possible value for such an additive constant is zero whenever proportionality between two of the variables is found. Given all the proportionalities there is only one form the gravitational equation can take. It is the one introduced by Isaac Newton and known today as Newton's Law of Universal Gravitation:

$$F = G m_1 m_2 / R^2$$

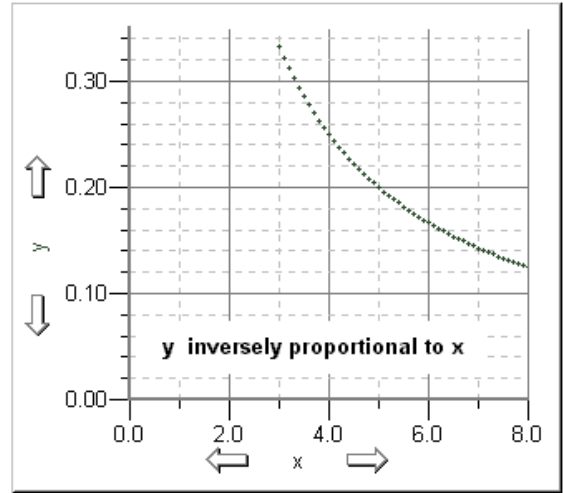
The proportionality constant, G, is known as the universal gravitational constant. Its universality is assured by the careful confirmation of all the proportionalities implied by this equation.

(BTW: the proportionality constant is one of a small number of critical universal constants and its precise experimental value remains a matter of utmost scientific importance. The best current value is $G = 6.672\,598\,5 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2$.)

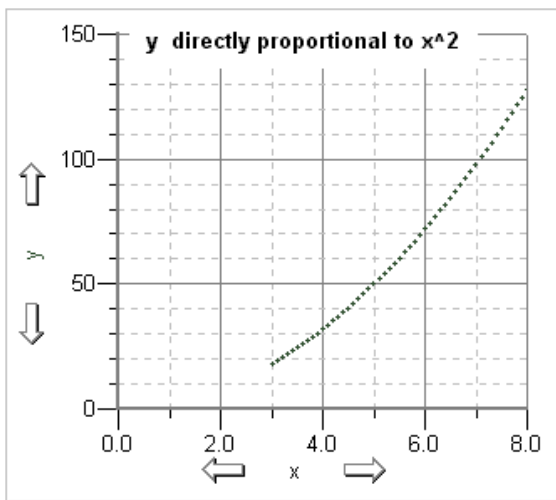
Direct Proportionality



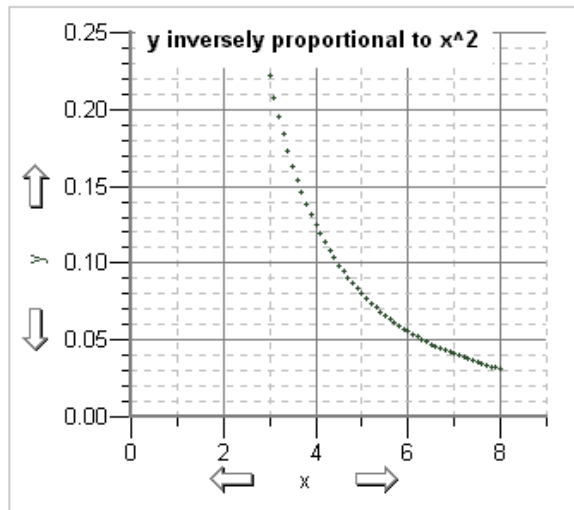
Inverse Proportionality



Directly Proportional to the Square



Inversely Proportional to the Square



Equations – We are not concerned here with any particular physics equation. What you should concentrate on is the form of a few equation types. You should recognize these types when you encounter actual physics equations that match these types.

Linear Equations – When the variables in an equation occur only to the first power, it is called a **linear equation**. When you graph pairs of variables from a linear equation, while the other variables are held constant, you will get a straight-line graph. Linear equations give us graphs with slopes that are determined by the other variables and the proportionality constant. Variables that control the slope must be held constant during an experiment; otherwise the slope will not be constant.

The classical straight-line equation has several forms on a graph of y vs x .

$y = mx$ m is the slope and the y -intercept is zero

$y = mx + b$ m is the slope and the y -intercept is b

$y - y_0 = mx$ m is the slope and the y -intercept is y_0

How do we actually use this in physics? Here is one simple example. An electric current, flowing through a resistor, experiences a voltage change. The voltage change, V volts, across the resistor, the current, I amps, flowing in the resistor, and the resistance of the resistor, R ohms, obey a relationship known as Ohm's Law,

$$V = IR$$

We could measure V and I under varying conditions using a single resistor with a constant resistance. A " V vs I " graph of the results of those measurements would plot as a series of points that lie along a straight line with a slope of R and a y -intercept of zero.

We analyze the data with two aims in mind: find the resistance of the resistor and check the data to make sure it does in fact obey Ohm's Law. We have a program that will allow us to find the best straight line through the origin. The slope of that line should be the resistance in a V vs I graph. (*What is the slope in a plot of I vs V ?*) If the data is good, $R = \text{slope of the line through the origin}$. We can use the same program to find the best straight line of the form $y = mx + b$. This line will have a slightly different slope, but if the data is good and our experimental circuit obeys Ohm's Law the two slopes should be very similar and the y -intercept on the second graph should be very close to zero. If the two slopes are very similar and the y -intercept is close to zero, we can have a high degree of confidence in the accuracy of our data and our experimentally determined resistance.

Quadratic Equations – When the variables in an equation occur only to the first and second power, it is called a **quadratic equation**. The graph of a quadratic equation is a parabola. When you construct a graph of your data you might see only part of the parabola, but don't be fooled.

One example you will encounter early in the course is the motion of a uniformly accelerated object. For linear motion subject to a constant acceleration along a straight line, which we will call the x-axis, let x_0 be the position at time zero and x be the position at another time, t . The equation describing the displacement, $x - x_0$, of such an object as a function of time, t , is

$$x - x_0 = v_0t + \frac{1}{2}at^2$$

A plot of x vs t will be a parabola.

The standard form of a parabola is

$$ax^2 + bx + c = 0$$

If we rearrange our displacement equation to put it in the standard form and set x equal to zero, we get

$$\frac{1}{2}at^2 + v_0t - x_0 = 0$$

If we have displacement vs time data in a table our first goal is to find the coefficients. We can use our graphing program for this task as well. To make the example more concrete, let's assume that the graphing program has provided us with the following numerical coefficients;

$$2t^2 + 3t - 5 = 0$$

We know from our general equation that the coefficient of t^2 equals one-half the acceleration. Therefore, $\frac{1}{2}a = 2$, therefore $a = 4$ and the acceleration = $a = 4 \text{ m/s}^2$.

We know from the general equation that the coefficient of t is the initial velocity, the velocity when our clock was reading 0.00 s, $v_0 = 3 \text{ m/s}$.

The equation also tells us that we made all of our measurements with x_0 held constant at the point labeled -5 m on our x coordinate axis. If our data was carefully measured, $c = x_0$ should be very close to the initial position we chose to use in the experiment.

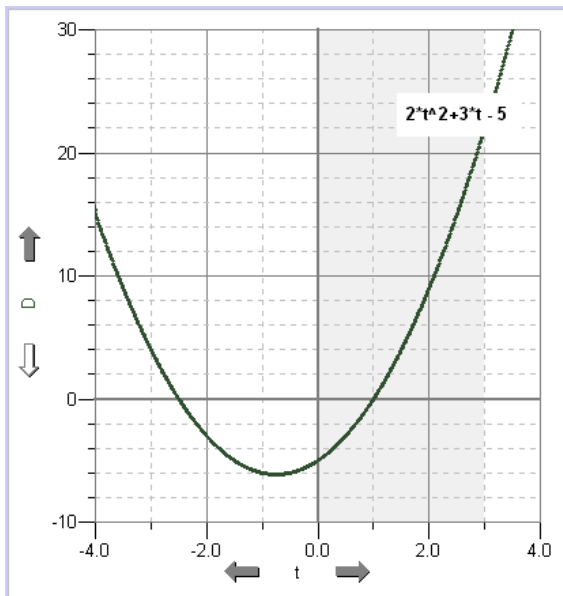
Once the graphing program has provided us with the necessary coefficients, we can use the resulting equation for other purposes. We might, for example, want to know how long it would take to travel a different distance. We have the coefficients of t^2 and t , and we could easily plug in a new value for x_0 . So the next question is how do you find the value of the time, t , needed to travel the new distance?

Solving a quadratic like this one for its independent variable, t in this case, means either using the quadratic formula or graphing the function to find the roots. The quadratic formula looks like this

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case we would replace the x with our t , the a with our $\frac{1}{2}a = 2$, b with our $v_0 = 4$, and the c with our x_0 . The plus-minus sign, \pm , reminds us that there are two answers. Yes, that's right – two answers. We could get two complex numbers as answers in a general math problem, but complex numbers do not refer to the real world. We expect to get two real numbers as answers. That is only possible if the expression under the radical is not negative, i.e. as long as $b^2 - 4ac \geq 0$. In the unlikely event that $b^2 - 4ac$ happens to equal zero, there will be only one solution and that point will correspond to the point at the vertex of the parabola. In fact this is one way to find the vertex. Vary c , our x_0 , until $b^2 - 4ac$ equals zero.

We'll use the equation for a 5 m displacement. Here is the graph of that quadratic equation. The roots appear to be -2.5 s and $+1.0$ s. (*The roots are the times where the curve crosses the horizontal axis.*)



We do not typically measure the negative times in most experiments. All the experimental data usually occurs only to the right of the zero time line.

If we use the quadratic formula to solve for the roots we get

$$t = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$$

$$t = \frac{-3 \pm \sqrt{9 + 40}}{4} = \frac{-3 \pm 7}{4}$$

Therefore, $t = +1.0$ s and $t = -2.5$ s.

The graph and the quadratic formula give us

the same roots.

Simultaneous Equations – In situations where we must find two unknown quantities at the same time we need two independent equations (*equations that are not simply different versions of the same equation; equations that cannot be turned from one into the other using the simple rules of algebra*). This is a common occurrence in physics. We will encounter it very early in the course during our work with force diagrams where we must find two of the three forces holding an object in equilibrium. We will encounter it again later in the course during our studies of torque and again in our study of electrical circuits.

Simultaneous equations can be solved classically using the rules of algebra and substitution and replacement techniques. Alternatively, there is a separate handout on the physics web site showing how to solve simultaneous equations using matrices on a TI calculator. This is a simple process and a very powerful technique.

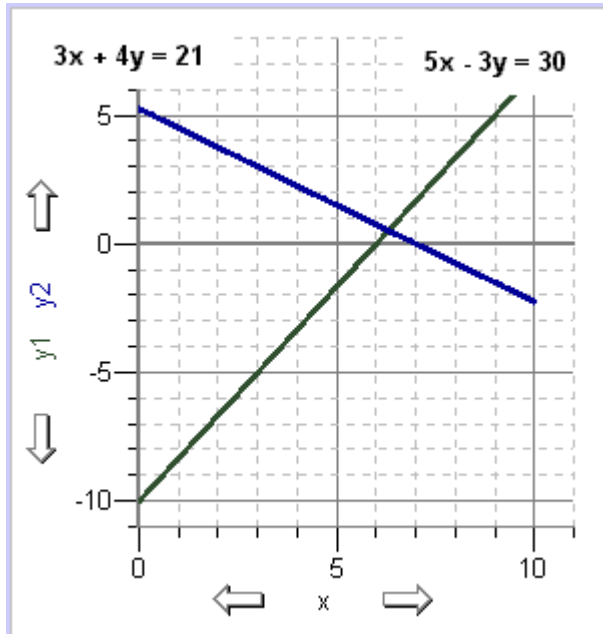
Equations that require use of these techniques have at least two unknowns and as many equation as there are unknowns. A simple case with two unknowns is

$$5x - 3y = 30$$

and

$$3x + 4y = 21$$

If x and y are both unknown, each of these linear equations taken independently has



many possible solutions. Just pick an x and solve for the corresponding y. What we want is the unique pair of numbers, one x and one y, that satisfy both equations at the same time. This is the same as finding that one unique point on the xy-plane where the two graphs cross. If the two graphs don't cross, the two lines are parallel, and the two equations do not share a solution. In all physically realizable situations the graphs do cross and there is a solution.

Whenever you solve two simultaneous equations, by whatever means, it is essential that you check the answers. This can be done by graphing both equations and examining the intersection. Generally

it is easier to check the result by plugging the x and y values back into both equations. In this particular example the solution is $x = 6.310345$, and $y = 0.517241$. Check them out to verify these results. Small errors due to calculator round off are acceptable. Make sure you check both equations.