Lesson 00-A5-Estimating and Approximation

Estimating and approximation have very different goals though they are often confused. We include them together here in part to highlight the differences.

Estimating – We perform an estimation either before or after a more precise calculation. The idea is to improve the likelihood that the careful analysis was performed correctly. To accomplish the check of the results all we need, sometimes, is what is called an order of magnitude calculation. If the order of magnitude calculation is close to the careful calculation we feel a justified confidence that the more careful calculation was performed correctly. The important characteristic of an estimate is that it should be quicker than the more careful calculation.

The procedure is fairly simple. Convert all numbers to scientific notation with only one non-zero digit. Imagine the following calculation needs to be performed.

$$[1{,}234{,}567 \times 447.23\times10^{-7} \times 0.000654\times10^{+13}] / [8003.77 \times 4.004\times10^{+6}]$$

If you perform this calculation as written on your calculator the answer is

11.267684

But how can you check this answer? One way is to repeat the calculation. If you do that, perform the individual steps in a different order. That may prevent you from repeating some typing error. The other way is to make a quick estimate. To make an estimate transform each individual term in the expression, as follows

[1E6 × 4E3×E–7 × 7E–4×E13] / [8E3 × 4E6]

which becomes

[1E6 × 4E–4 × 7E9] / [8E3 × 4E6]

[1×4×7 × E6 × E–4 × E9] / [8 × 4 × E3 × E6]

[28 × E11] / [32 × E9]

1 × E11 × E–9]

1 × E2 = 100

This is within one order or magnitude (*one factor of ten*) of the correct answer. This estimate supports the correctness of our calculation. In a case this extreme, it would still be wise to repeat the calculation just to be on the safe side.

Approximation – The idea here is to simplify the careful calculation. We are not trying to make a crude calculation as we did when estimating the answer. We are trying to simplify a complex derivation without introducing large errors into the calculation. We will accept small errors, however. We would prefer that the errors be smaller than the measurement errors inherent in the data.

You might think this is a waste of time since your calculator can handle any problems. However, your calculator is not your only tool for making calculations. Physics is also about deriving equations that accurately describe the relationships we observe in nature. Using these mathematical approximations in the derivation of an expression can get you over some exceedingly tedious mathematic hurdles. They lend simplicity to the mathematics and speed up the derivation. We will look at sample calculations below. That is done mostly to lend credibility to the approximations. The real value of these approximations is in simplifying derivations of the expressions you use to do your calculations.

The important thing to remember about approximations is that they are additional sources of error in our calculations. We neither use these randomly, without some thought to the consequences, nor often. Approximations are used best when used sparingly. There will be few opportunities to use these in your first physics course, but the opportunities will occur more frequently in later courses, like AP Physics C.

There are only a few approximations that are used with any regularity.

Newton's Binomial Expansion – The general form of Newton's Binomial theorem is

$$(a + b)^n = a^n + na^{n-1}b + \dots$$

for positive integer values of n, this expansion has a finite number of terms. For negative values of n it has an infinite number of terms. For purposes of using it in approximations, we are only interested in a special form of this series expansion. We only care about cases where $a = 1$ and b is a variable, say $b = x$. Furthermore, we are only interested in cases where $x \ll 1$. Because x is so small, the later terms where x gets raised to higher and higher powers are small enough to ignore. Typically, we only keep the first two terms of the expansion. That is the source of the error introduced with this method. We drop the later terms in the expansion.

The simplicity this approximation introduces has to be worth the risk of introducing this error; otherwise we should not use it. REMEMBER – this only works if we are sure that the variable represented by x is a very small number. If you are not sure it is small, then do not use this approximation.

Here are some reasonably common substitutions based on the truncated binomial expansion.

[general case, x<<1]	$(1+x)^n \approx 1 + nx$	$(1-x)^n \approx 1 - nx$
for n = 2	$(1+x)^2 \approx 1 + 2x$	$(1-x)^2 \approx 1 - 2x$
for n = 3	$(1+x)^3 \approx 1 + 3x$	$(1-x)^3 \approx 1 - 3x$
for n = ½	$(1+x)^{1/2} \approx 1 + \frac{1}{2}x$	$(1-x)^{1/2} \approx 1 - \frac{1}{2}x$
for n = –½	$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$	$(1-x)^{-1/2} \approx 1 + \frac{1}{2}x$

Try this example. Suppose you needed to take the square root of 1.000 000 001. Your calculator probably can't handle this. Mine reports the answer as 1.000 000 001. Even a rough approximation would be better than that. Write the problem in the form of a binomial.

$$(1 + 0.000\,000\,001)^{1/2} \approx 1 + \tfrac{1}{2}\cdot 0.000\,000\,001 \approx 1.000\,000\,000\,5$$

This at least makes sense. Surely the square root of any number larger than one will be between one and the number. You can also use the binomial approximation to recover the original number.

$$(1 + 0.000\,000\,000\,5)^2 \approx 1 + 2\cdot 000\,000\,000\,5 \approx 1.000\,000\,001$$

Other Truncated Series Approximations - Here are two additional approximations based on truncated series. These also require that x<<1 to work with a minimum of error.

$$e^x \approx 1 + x$$

$$\ln(1 \pm x) \approx \pm x$$

Thus,

$$e^{0.000\,000\,001} \approx 1 + 0.000\,000\,001$$

$$e^{-0.000\,000\,005} \approx 1 - 0.000\,000\,001 = 0.999\,999\,995$$

and

$$\ln(1.000\,000\,001) \approx +0.000\,000\,001$$

$$\ln(0.999\,999\,999) = -0.000\,000\,001$$

Trigonometric Approximations – There are three trigonometric approximations you need to remember. These work only under the following limitations.

- The angles must be entered in radians in all three approximations. If you ever use degrees with these approximations you will get large errors.

- The angles, when written in radians, must be small, $\theta << 1$.

Here are the three trigonometric approximations with examples.

$\sin\theta \approx \theta$

sin 0.001 = 0.001 (*true answer* = 0.000 999 999)
sin 0.01 = 0.01 (*true answer* = 0.009 999 833)
sin 0.1 = 0.1 (*true answer* = 0.099 833 417)
sin 0.2 = 0.2 (*true answer* = 0.198 669 331)

$\cos\theta \approx 1$

cos 0.001 = 1 (*true answer = 0.999 999 500)*
cos 0.01 = 1 (*true answer = 0.999 950 000)*
cos 0.1 = 1 (*true answer = 0.995 004 165)*
cos 0.2 = 1 (*true answer = 0.980 066 578)*

$\tan\theta \approx \theta$

tan 0.001 = 0.001 (*true answer* = 0.001 000 000)
tan 0.01 = 0.01 (*true answer* = 0.010 000 333)
tan 0.1 = 0.1 (*true answer* = 0.100 334 672)
tan 0.2 = 0.2 (*true answer* = 0.202 710 036)