Lesson 00-D-Vectors

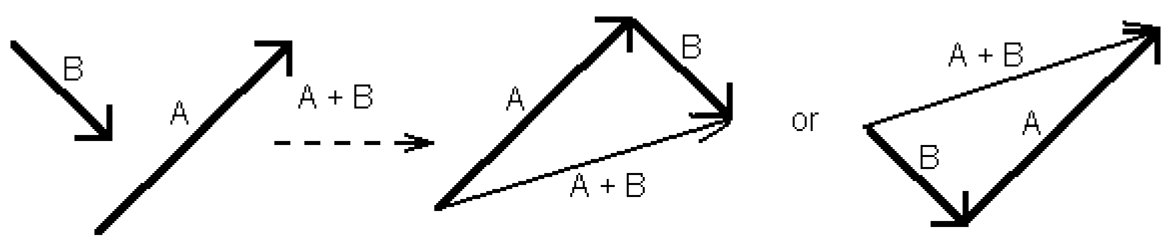
The mention of vectors in the last lesson was too brief for our needs. The topic there was trigonometric functions. They are useful for working with vectors to solve problems. Here you will be introduced to the vectors themselves. Vectors represent physical parameters that have both a magnitude and a direction. Examples include: Displacement, Velocity, Acceleration, Momentum, Force and others. We measure numerous quantities that are vectors. Trigonometric functions will make that job easier, but vectors have properties that depend on more than just the trigonometric functions.

A vector is a mathematical entity. Do not be fooled into thinking that it has a physical existence. It represents some physical parameter, but is not itself a physical object. The arrows we draw and the mathematical symbols we use are not the vector, either. These symbols only provide us with the means of representing vectors and working with vectors mathematically.

Representations of Vectors

You will never see a vector. Why? Have you ever seen a velocity? Think about it. You have seen things moving, things that have velocity. However, have you ever seen the velocity itself? No! We need visual and mathematical representations of vectors because we cannot see them. For this we use arrows and then we break those arrows into components using trigonometric functions. The representations embody all the important properties of vectors and serve as mathematical surrogates for vectors.

Here are two examples of vector representations. We use capital letters as names for vectors so they stand out in the text and so we can refer to them in equations.

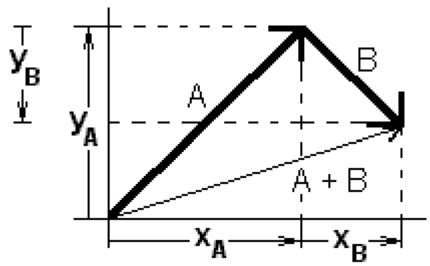


These arrows might represent any vector quantities with the same units. Let's assume they represent quantities called displacement. Basically, that is how far in one direction an object moves. If we push a box across the floor a distance A in the direction of the long arrow and then push it a distance B in the direction of the short arrow, it will have moved a displacement equal to the arrow marked **A** + **B**. The diagram shows that vector addition is independent of the order of the addition. The vector **A** + **B** has the same magnitude and direction no matter which order we use.

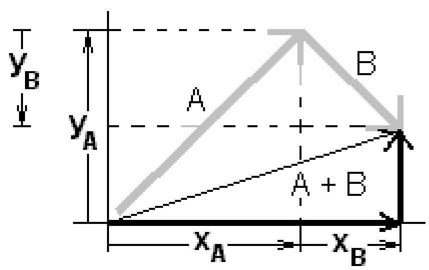
When the vectors are displacement vectors, vector addition describes the net result of consecutive movements of the object.

What if the vectors represent some other quantity, say velocity? First, let's ask this question: Can an object have more than one velocity at a time? There are two answers to this question.

The first answer is, "No." If you are talking about the object's total velocity, then it has only one. On the diagram this is the velocity vector labeled **A** + **B**.



The other answer is, "Yes." Suppose the object is in a large room the same shape and orientation as this page. You could just as well talk about the velocity of the object <u>in the direction of the top of the page</u> or its velocity <u>in the direction of the right side of the page</u>. As you can see, the object is moving toward both walls at the same time. Thus, it has two different velocities at the same time. One velocity points toward the top and the other toward the right side. The velocity toward the right side has a magnitude of (x_A + x_B). The velocity toward the top has a magnitude of (y_A – y_B). Note that the directions of the two "component vectors" is each toward its respective side.



We reserve the term components for special cases like this one where the two vectors are perpendicular to each other and parallel to the axis of our selected coordinate system. If you add the two components together you will get the original vector, **A** + **B**, with its original magnitude and its original direction.

So what about **A** and **B**? They might be perpendicular. They look close, but in this case they are not components of **A** + **B**. At least they are not components in the special sense in which we will be using it. **A** and **B** are merely two vectors that add up to our vector **A** + **B**. Components are a special pair; components are a pair of vectors that are perpendicular to each other AND parallel to the coordinates of our chosen coordinate system. With these criteria in mind, **A** and **B** could only be components of the vector **A** + **B** if they were perpendicular to each other and were each parallel to the axes of our chosen coordinate system. In this case **A** and **B** are neither perpendicular nor parallel to the coordinate axes.

The components of **A** + **B** appear in the last drawing, where you can see how they are created from the components of **A** and **B**.

Vector Addition

We've now seen two ways to represent vectors and two ways to accomplish vector addition. Lets formalize these observations.

First, representing the vectors:

Vectors as arrows – We can either draw the arrows as shown above or describe them in terms of there length, magnitude, and direction. The arrow is pretty obvious but not quantitative enough for mathematical purposes. To describe an arrow representing a vector all we need to do is specify its magnitude and direction. Here is an example:

$$33 \text{ m/s} \angle 145^\circ$$

This is the description of a velocity vector.

For our vectors **A** and **B**, we can write the general forms as follows.

$$\mathbf{A} = |\mathbf{A}| \angle \theta \quad \text{and} \quad \mathbf{B} = |\mathbf{B}| \angle \phi$$

The angles θ and ϕ are measured counterclockwise from the +x-axis. Clockwise angles can be used but in that case they are negative angles.

Vectors from their components – An even better way to describe a vector is in terms of its components. Referring to the vectors on the previous page, we can write the vectors **A** and **B** in terms of their components. Thus,

$$\mathbf{A} = x_A \boldsymbol{i} + y_A \boldsymbol{j} \quad \text{and} \quad \mathbf{B} = x_B \boldsymbol{i} + y_B \boldsymbol{j}$$

The symbols ***i*** and ***j*** represent what we call "unit vectors." Unit vectors have a length one unit and a direction parallel to the coordinate axes of our reference system. The unit vector named ***i*** points parallel to the x-axis and the unit vector named ***j*** points parallel to the y-axis.

These two ways of writing vectors are known as the polar form and component form, respectively. The polar form is the typical form that we get from making measurements. The component form is the form we typically use for doing vector mathematics.

Adding vectors in polar form

There is no easy way to add the vectors in polar form. The only way to add them is to draw them carefully and manually draw the result using what is called the head-to-tail method of vector addition. This is at best tedious and at all times error prone. The diagrams on page 2 show how the vectors **A** and **B** are added head-to-tail to obtain the resultant vector **A + B**.

Your only practical choice is to convert the polar forms to their equivalent component forms. We will learn a calculator technique you can use for this purpose in a few days. The basic principle is to use the following equations.

$$x_A = |\mathbf{A}| \cos\theta \quad \text{and} \quad y_A = |\mathbf{A}| \sin\theta$$

$$x_B = |\mathbf{B}| \cos\phi \quad \text{and} \quad y_B = |\mathbf{B}| \sin\phi$$

When we see vectors in polar form, our first task is to convert them to component form. The components of **A** and **B** can them be added. After you've added the component forms together, you will have to convert that answer back into the polar form of the answer. We will also learn a calculator technique for accomplishing that. The basic equations for the conversion from component form back into polar form are,

$$|\mathbf{A}| = (x_A^2 + y_A^2)^{1/2} \quad \text{and} \quad \theta = \tan^{-1}(y_A / x_A)$$

Then, **A** can be written as

$$\mathbf{A} = |\mathbf{A}| \angle \theta$$

Once we calculate A + B in component form, we can use the same conversion technique to find the polar form of the answer.

Adding vectors in component form

If you have the vectors represented in component form, all you have to do is add the x-components of each vector together. Then add the y-components of each vector together. The vector sum looks like this:

$$\mathbf{A} + \mathbf{B} = (x_A + x_B)\boldsymbol{i} + (y_A + y_B)\boldsymbol{j}$$

To convert this answer in component form into an answer in polar form, use the technique described above.

$$\mathbf{A} + \mathbf{B} = \{ (x_A + x_B)^2 + (y_A + y_B)^2 \}^{1/2} \angle \tan^{-1}\{ (y_A + y_B) / (x_A + x_B) \}$$