## Lesson 00-E-Dimensions or Units

When you determine the distance between two places or objects, you must use a unit of length measurement. That unit of measurement, unit for short, must travel with the measured number. Otherwise, the number is meaningless and useless. You must know whether the distance was measured in feet or meters, or miles or nanometers. Measurement includes the units and so must you.

Area, on the other hand, is measured as length times width. If you have a length in feet and a width in feet, you will get, upon multiplying the two together, an area in feet squared. We often say simply, "square feet". Again, this unit, square feet, must travel with the number through any calculations. The number without the units is useless. It makes a difference whether the area is measured in square inches or square miles and unless the unit is included there is no way to know which you are talking about.

Distances can be measured in many different units, but they all have one thing in common. They are measures of length. If we want to have a general discussion about measurement, we could just as well use a general symbol to stand in for all the length-measuring units. Let's use **[L]** to stand in for length in general. **[L]** is called the dimension of the unit.

Similarly for time and mass. Let's use **[T]** and **[M]** as general symbols for the dimension of these two measured quantities. We will use **[T]** and **[M]** to stand-in for all the time and mass discussing dimensions.

In these general terms, then, we can look at other measurable quantities to see what dimensions they will require. For example, in the case of area, we would say the dimension of area is **[L][L]** or $[\mathbf{L}]^2$. By logical extension, then the dimension of volume must be **[L][L][L]** or $[\mathbf{L}]^3$.

For a different example, what can we say about velocity? How fast an object moves depends on two things. How far it moves over what period. We can say that in general the dimensions of velocity are **[L]**/**[T]**. This could be feet per minute, meters per second, miles per hour or any combination of a length unit divided by a time unit.

This suggests a technique for using dimensional analysis to check the correctness of the equations we derive in the course of our investigations. Consider the following simple equation.

**A + B = C**

A, B and C represent physical quantities and have associated units. We don't yet have the numbers, but we do know some things about the dimensions. If the equation describes a valid relationship among the quantities, there are four rules governing the dimensions, which must be obeyed.

**Rules for Units in Equations**

1) ***The dimensions and units must be treated as algebraic quantities***. They must be added, subtracted, multiplied, and divided in accordance with the relationship defined by the equation. We can take roots and raise them to powers. When you work with an equation you must of course apply the same algebraic operations to both the numbers and their associated units and dimensions. In dimensional analysis we analyze the units or dimensions without worrying about the numbers.
(Examples: ft + ft = ft; ft X ft = $ft^2$; ft x ft x ft = $ft^3$; $ft^2 \div ft^3 = ft^{-1}$; $(ft^2)^{½}$ = ft; etc.)

2) ***Only the quantities having the same dimensions can be added together***. You probably already know that we cannot add apples to oranges or feet to meters. Those give us fruit salad and nonsense, respectively. But this rule is even more general. This rule says that you cannot add any type of different dimensions. It says you cannot add lengths to times or masses to distances, meters to seconds, kilograms to seconds, etc.

3) **For an equation to have any chance of being valid, the dimensions on both sides of the equal sign must be the same**. If the dimensions are the same, it does not prove the equation is correct; only that it might be correct. On the other hand, if the dimensions do not match, then the equation cannot possibly be correct. If that happens you need to find a new equation.

The fourth rule is a bit ahead of us, but you will need it later.

4) **You cannot use mathematical functions to operate on collections of physical quantities unless the collection is unitless**. The mathematical functions you will most often see in this course are sin, cos, tan, exp ($e^x$), log and ln. These mathematical functions are often used but only on collections of numbers that have no dimensions. We will discuss this rule later in the course when it becomes relevant.

If you study the relationship among a number of measured quantities and come up with an equation that satisfies all these rules, then your equation may be correct. There is no guarantee that it is correct, but it does have a chance of being correct.

## Evaluating the Dimensions of an Equation

You will shortly begin a series of investigations to test and verify the following equation.

$$x - x_0 = v_o t + \tfrac{1}{2}at^2$$

This equation says that the displacement of an object ($x - x_0$) from its initial position ($x_0$) to its final position at a later time (x) equals the initial velocity ($v_o$) times the time it took to make the trip (t) plus one-half the acceleration (½a) times the time it took to make the trip squared ($t^2$). This is quite a mouthful. To simplify the analysis, let's analyze only the dimensions.

On the left side of the equation we have two distances. The distances are understood to be measurements of the shortest distance from the location to the origin of our coordinate reference frame in each case. The dimension of distance or displacement is **[L]**. The difference of two distances or displacements is still **[L]**. Therefore, on the left side of the equation the dimension is **[L]** – **[L]** = **[L]**.

On the right side of the equation the situation is a little more complicated. As you can see it consists of two terms. According to our rules we can only add these two terms together if they both have the same dimensions. Furthermore, because the left side has dimension **[L]**, each of the terms on the right must have the dimensions of **[L]**. They both need to have the same dimension as the other side of the equation. Otherwise the equation is certain to be invalid.

Thus, we need to check each term separately to make sure it has dimension **[L].**

Start with $v_o t$. Velocity has dimensions of **[L]**/**[T]**. Time has dimension **[T]**. Multiply these together and you get

$$[L]/[T] \times [T] = [L][T]^{-1} \times [T] = [L]$$

In the second term, $\tfrac{1}{2}at^2$, the coefficient ½ has no units. The acceleration, a, has dimensions of $[L]/[T]^2$. Acceleration is change in velocity per unit of time; the one you'll be using most often is meters per second squared, or $m/s^2$. The dimension of $t^2$ is $[T]^2$. Multiplying these together we get the dimensions of the second term as follows.

$$[L]/[T]^2\ [T]^2 = [L][T]^{-2} \times [T]^2 = [L]$$

So, adding the dimensions on the right side together we get **[L]** + **[L]** = **[L]**.

Our equation might be right. Dimensional analysis says that the dimensions are good. The theory could still be wrong. (*Done get confused here. If you add 3 ft to 4 ft you get 7 ft.*)