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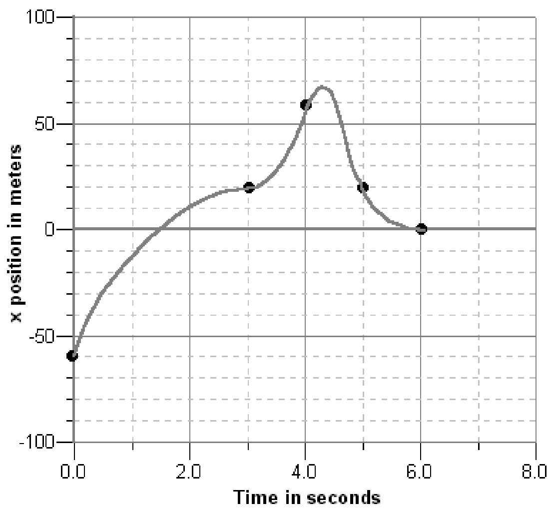
Physics: An Incremental Development, John H. Saxon, Jr.
Lesson 10 - Motion Graphs

The title refers to graphs of position, velocity or acceleration vs time. The black dots on these introductory graphs show the times where position, velocity or acceleration is known. Curved lines between the show approximately how the values were changing between the data points. In these early examples, the average values, "average velocity" and "average acceleration" will be computed between data points, only. Soon we will be able to work with the lines between the data points, too, but not immediately.

All the data in these graphs is reported relative to an arbitrary zero of time. Position, velocity and acceleration are measure relative to pre-selected reference frame. Generally, we will assume that a fixed point on the Earth represents the origin of that reference frame.

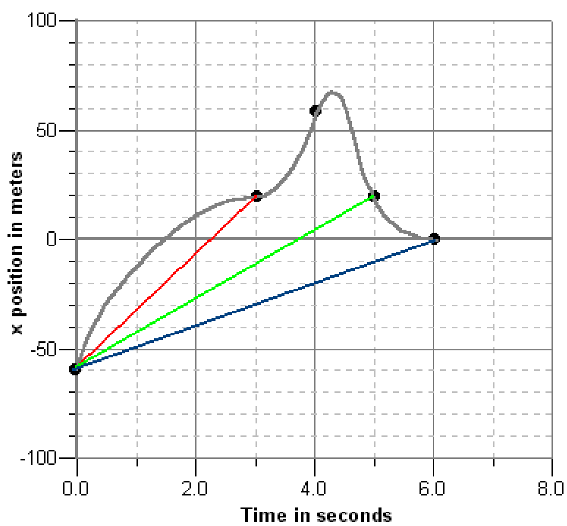
Position vs Time Graph



The value of the x-coordinate is simply read off the scale at various times as the object moves along the x-axis. The points on the graph show the measurements and the gray lines show the approximate locations of the object between the measurements.

The average velocity between any two data points can be determined from the slope of the straight line connecting the points.

For example:



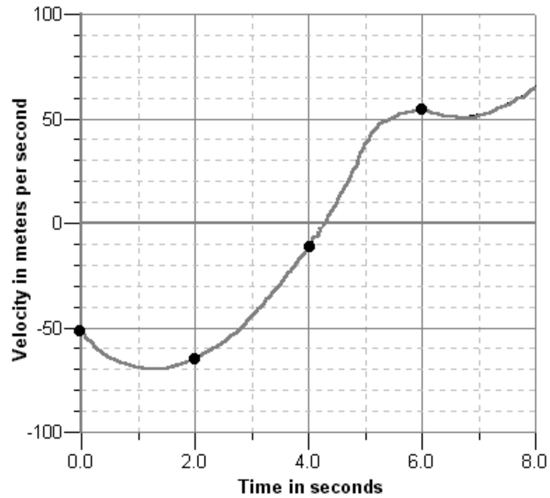
The average velocity between time = 0 s and time = 3.0 s is the slope of the red line in the figure on the right.

The average velocity between time =0 s and time = 5.0 s is the slope of the green line in the figure on the right.

The average velocity between time = 0 s and time = 6.0 s is the slope of the blue line in the figure on the right.

We know the slope of this line is a velocity, because slope equals $\Delta y/\Delta x = (x_f - x_i) / (t_f - t_i) = \Delta x/\Delta t$ = average velocity. This is a general result; **the slope on a position vs time graph is average velocity.**

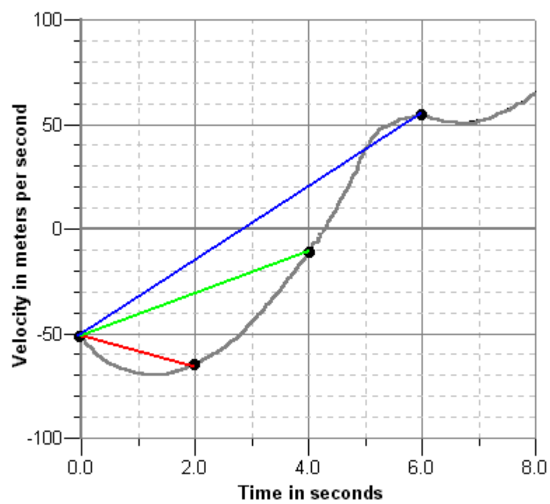
Velocity vs time graph



The value of the velocity is simply read off the scale at various times as the object moves along any straight-line axis. The points on the graph show the measurements of velocity and the gray lines show the velocity of the object between the measurements.

The average acceleration between any two data points can be determined from the slope of the straight line connecting the points.

For example:



The average acceleration between time = 0 s and time = 2.0 s is the slope of the red line in the figure on the right.

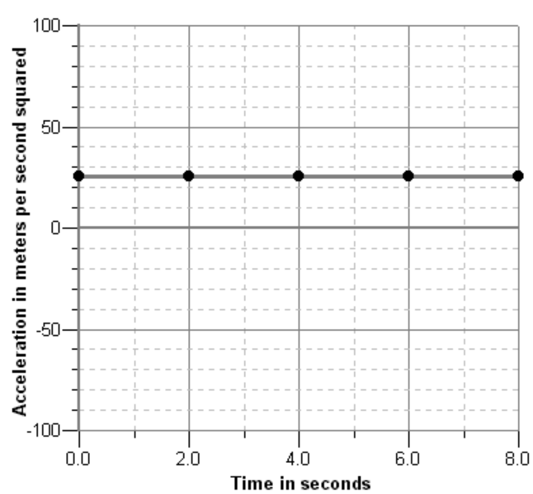
The average acceleration between time = 0 s and time = 4.0 s is the slope of the green line in the figure on the right.

The average acceleration between time = 0 s and time = 6.0 s is the slope of the blue line in the figure on the right.

We know the slope of this line is an acceleration, because slope equals $\Delta y/\Delta x = (v_f - v_i) / (t_f - t_i) = \Delta v/\Delta t$ = average acceleration.

This is a general result; **the slope of a velocity vs time graph is average acceleration.**

Acceleration vs time graph



We are most interested in cases where a constant force is applied to an object. This produces constant acceleration. We will not try, in this course, to analyze any cases where the acceleration changes with time.

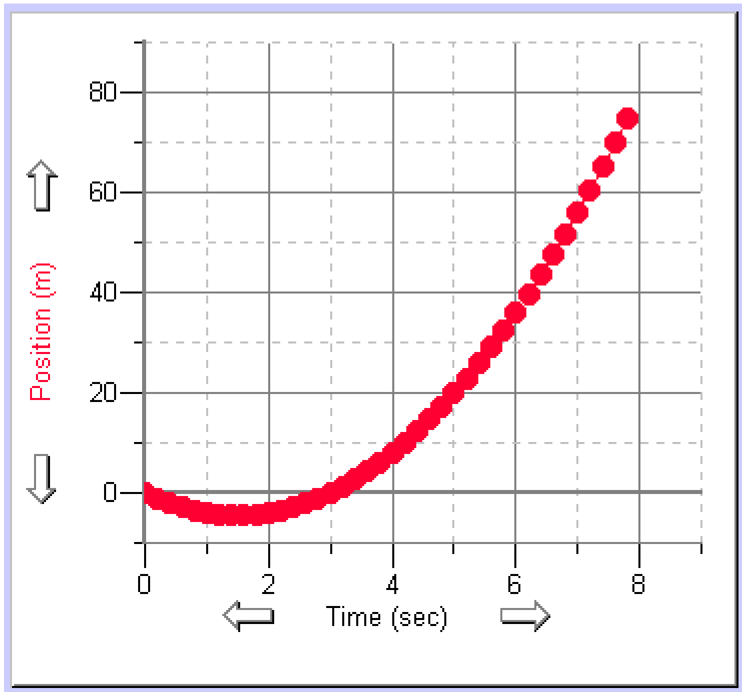
A typical constant acceleration graph is shown in the figure on the right.

Finding one graph from another

Have you noticed that the position graph gives you velocity, and that the velocity graph gives you acceleration? All these graphs are related to each other!

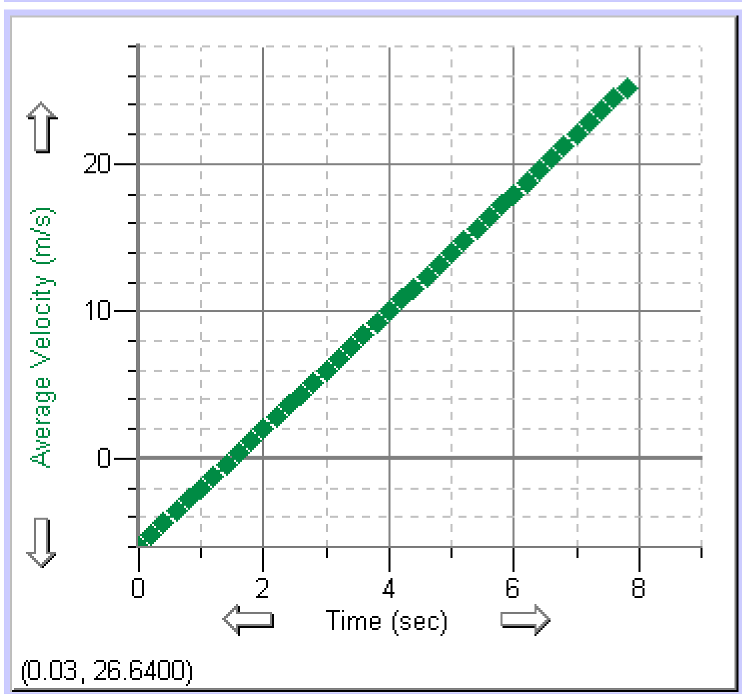
If we start with the position vs time graph, we can create a velocity vs time graph. Once we have a velocity vs time graph, we can create an acceleration vs time graph. The key is to use the slope of the line in one graph to generate a data point in the next graph. To make this work, we need data points in the first graph that are very close together. Then we will get the slope between successive points and plot these slopes in the next graph.

Here is the first graph. It shows position vs time. The data points are plotted every one-fifth of a second. I've chosen this graph because it is the position vs time graph of an object subjected to a constant force. The data points follow a parabola.



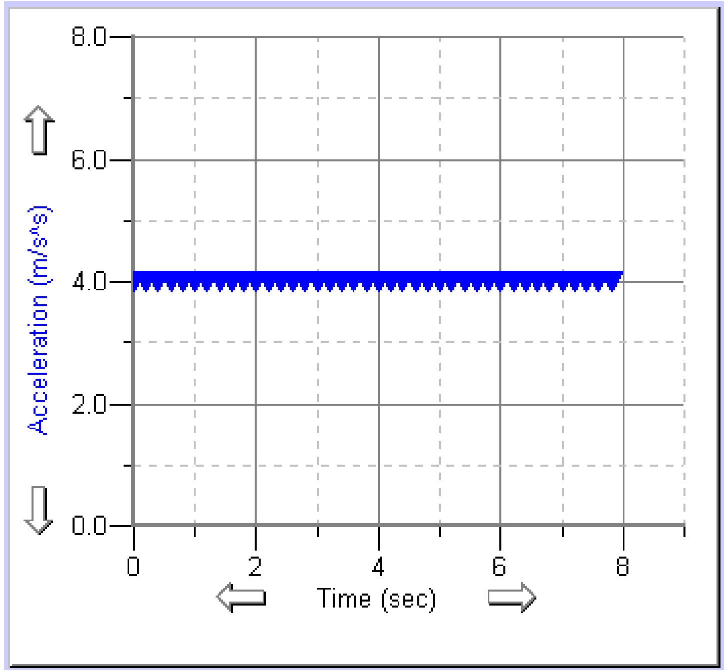
In this example the object appears to have a negative initial velocity. Notice that the object moves from the origin into negative position values before turning around and heading back into the region of positive positions.

The next task is to find the slope of the line between each consecutive pair of data points. That will give is the average velocity during each one-fifth of a second. To get the velocity graph, plot those average velocities vs time. That produces the graph at the right.



This graph is a straight line. The velocity is increasing steadily. That is the mark of a constant acceleration system. That means constant force, too.

The next task is to find the slopes between each successive pair of data points to find the acceleration.



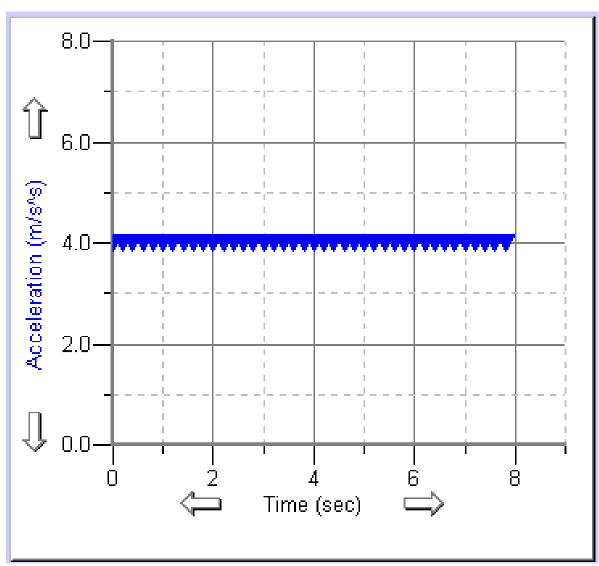
Here is the graph of the slopes from the velocity vs time graph. It shows that the acceleration is the same at all times. We knew it was a constant force situation so we should not be surprised that we find a constant acceleration.

Taking the slopes over very small intervals was the key to converting the information in one graph into the data for the next graph.

In the limit of infinitely small intervals, this process is known as taking the derivative. We will discuss the derivative indirectly in Lesson 13 and more thoroughly in Lesson 22.

Finding one graph from another - Redux

If taking the slope is the key to converting from position, to velocity, to acceleration, what operation acts to convert these graphs in the opposite direction? What operation is the opposite of taking the slope? While it is not immediately obvious, there is such an operation. It involves taking the area between the graph and the x-axis. Unfortunately, to go from acceleration, to velocity, or from velocity to position, we also need to know one more thing. This is called the initial condition.



We will use the same three graphs just discussed, starting with the acceleration graph. Check a few samples of the area under the graph at intervals; say every two seconds to see how this works.

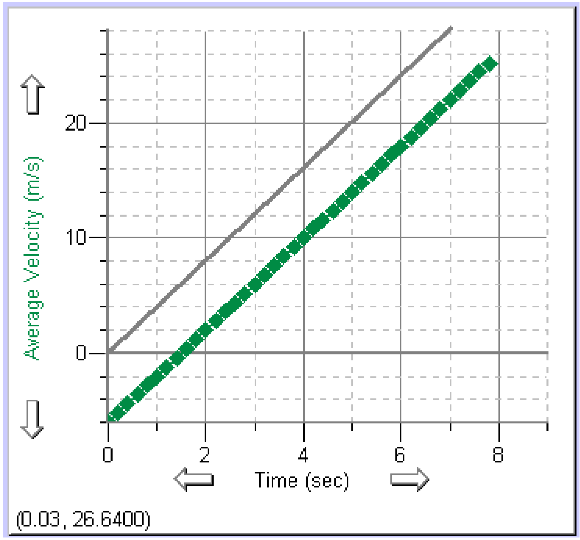
From t = 0 to t = 2 s, A = 8.0 m/s

From t = 0 to t = 4 s, A = 16.0 m/s

From t = 0 to t = 6 s, A = 24.0 m/s

From t = 0 to t = 8 s, A = 32.0 m/s

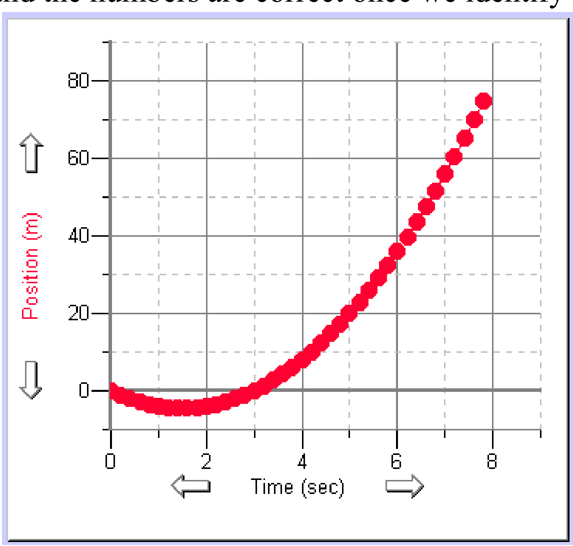
If you plot the points (2,8), (4,16), (6,24), and (8,32), you get a line with the same slope as our velocity vs time graph but all the points and the line itself are 6 units above the correct line. What we have found is a line, exactly like the correct line, but with a y-intercept of zero. (See the gray line.) This is where the initial condition comes in. The initial condition tells us what the true intercept must be. As you can see by looking at the original graph, the y-intercept at t = 0 is –6.0.



The points we should be plotting are, therefore, (0,-6), (2,8-6), (4,16-6), (6,24-6), and (8,32-6). If you plot all the data points this way you get the original graph back, as shown in the figure on the right.

In the limit of infinitely small intervals, taking the area between the line and the horizontal axis is also known as finding the integral. The key to evaluating integrals, often, is finding the initial condition. Experimentally, this is usually quite easy. All you have to do is make a note of where everything is positioned and how it is moving when you designate time zero.

Notice the unit of area in the acceleration graph. They are $m/s^2 \cdot s = m/s$. The units of area on that graph are correct for velocity and the numbers are correct once we identify the value of the initial condition.



Will the same trick work to recreate the position graph from the velocity graph. Notice that the velocity graph intersects the x-axis at 1.50 sec and that the position graph reaches its lowest point at 1.50 sec. At t = 1.5 sec the object comes to rest for an instant. Before that is was moving with a negative velocity. After that it is moving with a positive velocity.

A careful reading of the position graph shows that at 1.50 sec the object is at position –4.50 meters. On the velocity graph there is a small triangle between the x-axis that the graph. It extends from t = 0 s to t = 1.5 s. The area of that triangle is

$$\text{Area} = \tfrac{1}{2}\cdot(-6.0\ \text{m/s})\cdot(1.5\ \text{s}) = -4.5\ \text{m}$$

Which is exactly the value for the position we were looking for. This is not a fluke.

We can see from the position graph that the position is zero at 3.0 s. What about the area on the velocity graph between the graph and the x-axis for the interval t = 0 s to t = 3.0 s? This time interval includes two small triangles with identical areas. The only difference between the two triangles is that one is below the x-axis and has a negative area, while the other triangle is above the x-axis and has a positive area. Since the two areas are equal with opposite signs, when we add them together the sum is zero. Again the area gives us an exact calculation of the position.

In the limit of infinitely small intervals, taking the area between the line and the horizontal axis is also known as finding the integral. The key to evaluating integrals, often, is finding the initial condition. Experimentally, this is usually quite easy. All you have to do is make a note of where everything is positioned and how it is moving when you designate time zero. In this case our initial condition is that at time t = 0 s the position was at 0.0 m. Therefore no correction is needed in this example.

This is working so well, let's try one more little test. Find the position at t = 8.0 s.

To do that find the change in position between t = 0.0 s and t = 8.0 s. There are two triangles. The first is the one we found earlier, namely; $Area_1 = -4.5$ m. The second triangle starts at t = 1.5 s and extends to t = 8.0 s. Its area is

$$Area_2 = \frac{1}{2}\cdot(26.0\ m/s)\cdot(6.5\ s) = 84.5\ m$$

Which means that the total area over the entire interval is

$$Area = Area_1 + Area_2 = -4.5\ m + 84.5\ m = 80.0\ m$$

Again we get exact agreement with the position graph.

Summary

This mathematical tricks of using slopes (derivatives) and areas between the graph and the x-axis (integrals) is extremely useful in working with motion graphs as well as in many other aspects of physics and the physical sciences. This has been merely a demonstration of the concept. Later we will learn how easy it is to calculate derivatives and integrals more directly without appealing to a graph to guide us through the process.

We have learned that we can use the slope (derivative) to convert position vs time graphs to velocity vs time graphs. We have learned that we can use the slope (derivative) to convert velocity vs time graphs to acceleration vs time graphs.

We have learned that we can use the area between the graph and the x-axis (integral) to convert acceleration vs time graphs to velocity vs time graphs. We have learned that we can use the area between the graph and the x-axis (integral) to convert velocity vs time graphs to position vs time graphs. We have learned the to evaluate the integral we also need information about the initial condition on the graph we seek to create.