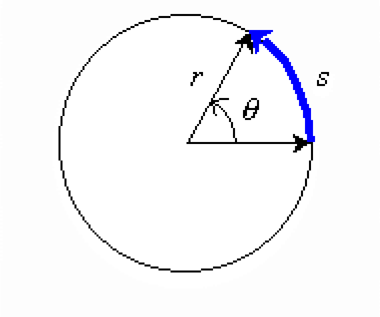
Chapter 25 - Measure in Radians

In mathematics and in physics it is always preferable to measure angles in radians rather than degrees. Radians are defined by referring to the following diagram. Here r is the radius of the circle, and as we swing the radius through an angle θ it traces out an arc length s, shown in blue. The arc length is not a straight-line distance; it is the curved distance along the circumference of the circle. The angle θ is measured in radians according to the following definition:

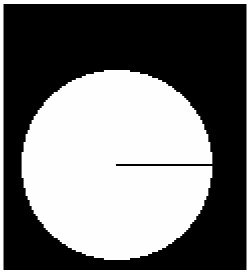


Definition of angle in radians

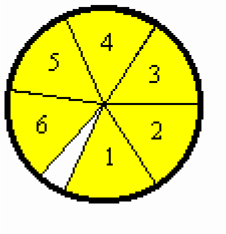
$$\theta = \frac{s}{r}$$

In words, the measure of an angle in radians is the ratio of the arc length swept out by that angle to the radius of the given circle. The size of the radius does not matter because the ratio of r to s will always be the same for a given angle. If we increase r, s increases by the same ratio.

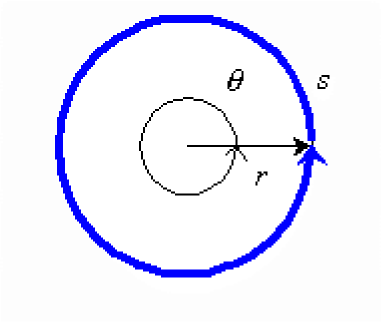
This means that one radian is the angle formed by an arc length that is equal to the radius:



To see how many radians make up a full circle, we can see how many 1-radian "pie-slices" it takes to complete a cycle. You can see from the diagram that the answer will be a little bit more than 6: the actual value is 2π radians, exactly.



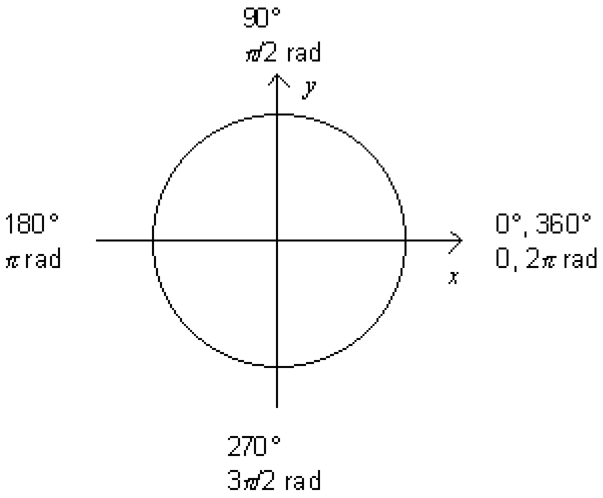
We can make this exact calculation by using the definition of the angle measured in radians, $\theta = s/r$. Let the angle θ swing through a full 360° circle, which means that the arc length will be one complete circumference of the circle. The circumference of a circle is given by the formula circumference = $2\pi r$, so the radian measure of the angle becomes



$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$$

Thus there are 2π radians in a circle (2π is about 6.28). Note that one radian is big—there are 360 degrees in a circle but only a little more than 6 radians. One radian turns out to be about 57.3°. This is somewhat awkward to work with, so it is usually best to represent radians as multiples of π. For example, 180° = π rad, 90° = π /2 rad, 45° = π /4 rad, etc.

The following diagram shows some common angles in standard position. Standard position angles are measured counter-clockwise from the positive x-axis.



Angular Velocity

When working with rotating systems, it is often easiest to work with angular velocity, **ω**, rather than with linear velocity, **v**. While the linear velocity can be measured in meters / second, as long as we measure the distance traveled along the arc, that distance is seldom easy to measure. It usually is much easier to measure the angle through which the object has rotated, and that means it will be easier to measure the velocity and the distance traveled in some angular measure. Because we will be measuring the angles in radians, and because the angle in radians is the ratio of two distances, we can convert between the two velocity measurements. Thus

$$\mathbf{v} = r\omega$$

And we can easily calculate the linear velocity even when we do not measure it directly. The true value of the definition of radian measure is that it provides us with the ability to interchange the two velocity measures. That interchangeability is our guarantee that the two systems we will be using, the linear and the angular, are consistent with each other. If we were trying to use angular measure in degrees the relationship would appear to be much less tightly linked. Because velocity and angular velocity are linked by our definition of the radian, we can be sure that all the work we have done with Newton's Laws of Motion will carryover to rotational system without fear that we are introducing some new, hidden difficulty.

We could, in fact, work out the details of the motion of all rotating systems using purely linear arguments and equations. It would be messy, but it could be done. We choose, instead, to use equations designed for angular motion because they are simpler to apply, not be cause rotation is fundamentally different from linear motion. Everything we've learned or will learn about position, velocity, acceleration, momentum, and energy is applicable to rotating systems. By using the definition of the radian presented here, we have insured that the equations will even take the same form. The symbols will be different, **ω** instead of **v**, for example, but the form of each equation will be the same as it is in the linear systems we study.

To show you how this works, we introduce a few of the most basic equations here alongside the linear forms we used to study linear motion; just so you can see the comparison.

Linear Motion Equations	Angular Motion Equations
Distance = rate x time	angular distance = angular rate x time
$d = v\ t$	$\theta = \omega\ t$
$x - x_0 = v_0 t + \frac{1}{2}at^2$	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$ where α = angular acceleration
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
Linear Momentum = $p = mv$	Angular momentum = $L = I\omega$ Where I = moment of inertia

Analogous equations can be written for all the other equations of motion that we have or will see.